

ELLIPTIC BOUNDARY PROBLEMS WITH RELAXED CONDITIONS

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Dedicated to M.Sh.Birman on the occasion of his 70th birthday

ABSTRACT. We use the Fredholm property of the operator associated with an elliptic boundary problem on a compact manifold with boundary to prove solvability results for a “relaxed” problem where the equation or the boundary conditions are “relaxed” (not required to hold) on a non-empty open set. We also formulate a generalization of the Fredholm property to the case when the problem is considered on a non-compact manifold M with a free action of a discrete group Γ such that M/Γ is compact. This is done by applying the Fredholm theory of M.Breuer in an appropriate von Neumann algebra to the corresponding operators which are considered in uniform Sobolev L^2 spaces. We deduce existence of infinite-dimensional space of solutions of the problem if zero conditions are imposed outside of an open Γ -invariant subset of the boundary.

INTRODUCTION

In this paper we study explicit existence results for “relaxed” elliptic boundary problems on a compact manifold with boundary or on a regular covering of such a manifold (in the non-compact case the existence is studied in the uniform Hilbert scale of the Sobolev spaces). Here “relaxing” of the problem means that we allow the equation or the boundary conditions not to hold on a non-empty open subset. For the homogeneous problem (with vanishing right hand side and boundary conditions) this immediately implies the existence of an infinite-dimensional space of solutions. The main argument is a simple linear algebra inequality on the dimension of the intersection of two linear subspaces of a given linear space (in the covering case the von Neumann dimension is used).

The case of the non-homogeneous problem is subtler. Here we need a duality argument, which reduces the problem to a uniqueness statement. The latter can be deduced in our case from the uniqueness of solution in the Cauchy problem.

Relaxed problems are usually a natural reformulation of control problems (we use the piece of the domain or the boundary where the conditions are relaxed, to control the right hand sides so as to assure the fulfillment of the solvability conditions).

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A different type of relaxed problem was mentioned in [GS]: it was deduced there from a general Riemann-Roch theorem that the Cauchy problem for the Poisson equation $\Delta u = f$ can be solved if the exact vanishing of the Cauchy data (on an arbitrary compact set of the Lebesgue measure 0) is replaced by an arbitrary number of linear orthogonality conditions (to measures). This result served us as a source of inspiration.

We treat the compact case in Section 1 and the covering case in Section 2. We also formulate some open problems in Section 3.

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1. EFFECTIVE SOLVABILITY RESULTS ON COMPACT MANIFOLDS

Let M be a compact C^∞ -manifold with C^∞ -boundary ∂M . Denote $n = \dim_{\mathbb{R}} M$. For simplicity we will always assume that M is connected. Consider an elliptic boundary value problem on M :

$$(1.1) \quad \begin{cases} Au = f & \text{on } M, \\ B_j u|_{\partial M} = \phi_j, & j = 1, \dots, r. \end{cases}$$

Here A is a differential operator of order m with C^∞ coefficients on M , B_j are differential operators (also with C^∞ coefficients) of orders $m_j < m$ near ∂M . The ellipticity of the problem (1.1) means that the operator A is elliptic everywhere on M and the Schapiro–Lopatinskii condition is satisfied at all points of ∂M .

Let us recall that the ellipticity of the operator A means that its principal symbol $a_m = a_m(x, \xi)$ does not vanish for all $x \in M$ and all $\xi \in \mathbb{R}^n \setminus \{0\}$. For $n \geq 3$ this implies that the operator A is *properly elliptic* i.e. m is even and for every $x \in M$ and every $\xi' \in \mathbb{R}^{n-1}$ the equation $a_m(x, \xi', z) = 0$ with respect to $z \in \mathbb{C}$ has exactly $m/2$ roots (multiplicities counted) in the upper half-plane $\{z \mid \text{Im } z > 0\}$ (it follows that all other $m/2$ roots are in the lower half-plane $\{z \mid \text{Im } z < 0\}$). If $n = 2$ this is not necessarily true but for the sake of simplicity we will assume proper ellipticity in this case as well.

Let us recall the Schapiro–Lopatinskii condition. Let $p \in \partial M$ be fixed. Choose local coordinates x_1, \dots, x_n near p so that p becomes origin and a neighborhood of p becomes

$$\{x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n) \mid x_n \geq 0\}.$$

Let $a_m(x, \xi) = a_m(x, \xi', \xi_n)$ be the principal symbol of A , $b_{m_j}(x, \xi', \xi_n)$ the principal symbol of B_j . Fixing arbitrary $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ consider the following boundary value problem on $\mathbb{R}_+ = \{t \mid t \geq 0\}$:

$$(1.2) \quad \begin{cases} a_m(0, \xi', \frac{1}{i} \frac{d}{dt})u(t) = 0 & \text{on } \mathbb{R}_+, \\ b_{m_j}(0, \xi', \frac{1}{i} \frac{d}{dt})u(t)|_{t=0} = \phi_j, & j = 1, \dots, r, \end{cases}$$

where $\phi_j \in \mathbb{C}$. So this problem is obtained from (1.1) by taking $f = 0$, freezing coefficients of A and B_j at p , replacing A and B_j by their higher order parts and making Fourier transform in x' . The Schapiro–Lopatinskii condition at p says that the problem (1.2) should be uniquely solvable for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ and all $\phi_1, \dots, \phi_r \in \mathbb{C}$.

It is well known that for a properly elliptic A the ellipticity of the problem (1.1) implies that $r = m/2$.

We can consider the following operator associated with the boundary value problem (1.1):

$$(1.3) \quad \mathfrak{A} : H^s(M) \longrightarrow H_s(M, \partial M) \\ u \longmapsto (Au, B_1 u|_{\partial M}, \dots, B_r u|_{\partial M}),$$

where

$$H_s(M, \partial M) = H^{s-m}(M) \times \prod_{j=1}^r H^{s-m_j-1/2}(\partial M),$$

$s \geq m$, and the spaces $H^t(M)$, $H^t(\partial M)$ are the Sobolev spaces from the standard Hilbert scales on M and ∂M respectively, i.e. scales based on $L^2(M)$ and $L^2(\partial M)$ (see e.g. [LM] for the definition). Then \mathfrak{A} is a bounded linear operator.

The main classical theorem of the theory of elliptic boundary value problems states that the operator \mathfrak{A} is Fredholm (in fact the ellipticity condition is equivalent to this Fredholm property). This means that the algebraic kernel and cokernel of \mathfrak{A} are finite-dimensional (which implies that the image of \mathfrak{A} is closed), or equivalently that $\dim \text{Ker } \mathfrak{A} < \infty$, $\text{Im } \mathfrak{A}$ is closed and $\dim \text{Ker } \mathfrak{A}^* < \infty$. In other words the solutions of the homogeneous problem (with $f = \phi_1 = \dots = \phi_r = 0$) form a finite-dimensional space, and the problem (1.1) can be solved if the right hand sides f, ϕ_1, \dots, ϕ_r satisfy a finite number of orthogonality conditions.

The classical examples of elliptic boundary value problems are the Dirichlet problem

$$(D) \quad \begin{cases} \Delta u = f & \text{on } M, \\ u|_{\partial M} = \phi, \end{cases}$$

and the Neumann problem

$$(N) \quad \begin{cases} \Delta u = f & \text{on } M, \\ \frac{\partial u}{\partial \nu}|_{\partial M} = \phi, \end{cases}$$

where M is a connected domain with a smooth boundary in \mathbb{R}^n , Δ is the standard Laplacian on \mathbb{R}^n (i.e. $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$), ν is an interior unit normal to ∂M . The Dirichlet problem is uniquely solvable (i.e. the corresponding operator (1.3) is invertible). As to the Neumann problem, both kernel and cokernel for the corresponding operator \mathfrak{A} are one-dimensional; namely, the kernel consists of constants and the problem is solvable for the right hand sides f, ϕ if and only if the following solvability condition is satisfied

$$(1.4) \quad \int_M f dx + \int_{\partial M} \phi dS = 0,$$

where dx is the Lebesgue measure on M , dS is the Riemannian density on ∂M .

A question which was often asked during early stages of development of the theory of elliptic boundary value problems is whether the Fredholm property can be used towards establishing some explicit solvability results. Below we give two examples when an explicit solvability type statement can be verified without any additional requirements on the problem. To do this we relax the equation and the boundary conditions in (1.1) by requiring them to take place only on $M \setminus U$ and $\partial M \setminus V$ where U, V are open subsets in $M, \partial M$ respectively, and one of the sets U, V is non-empty. For simplicity we will always assume that U, V have smooth boundaries, otherwise we can always shrink them a little to open sets with smooth boundaries. This does not lead to any loss of generality in the problems which we are going to consider. We are coming to the following relaxed problem

$$(1.5) \quad \begin{cases} Au = f & \text{on } M \setminus U, \\ B_j u|_{\partial M \setminus V} = \phi_j, & j = 1, \dots, r. \end{cases}$$

Let us consider also the corresponding operator

$$(1.6) \quad \begin{aligned} \mathfrak{A}_{U,V} : H^s(M) &\longrightarrow H_s(M \setminus U, \partial M \setminus V), \\ u &\longmapsto (Au|_{M \setminus U}, B_1 u|_{\partial M \setminus V}, \dots, B_r u|_{\partial M \setminus V}). \end{aligned}$$

where $H_s(M \setminus U, \partial M \setminus V)$ is defined as in (1.3) i.e.

$$H_s(M \setminus U, \partial M \setminus V) = H^{s-m}(M \setminus U) \times \prod_{j=1}^r H^{s-m_j-1/2}(\partial M \setminus V).$$

Here the spaces $H^r(M \setminus U)$, $H^r(\partial M \setminus V)$ (with $r \geq 0$) can be described e.g. as spaces of functions which admit extensions to the functions in $H^r(M)$, $H^r(\partial M)$ respectively.

Then we have the following elementary

Proposition 1.1. *If one of the sets U, V is non-empty then $\dim \text{Ker } \mathfrak{A}_{U,V} = \infty$, i.e. the problem (1.5) with $f = \phi_1 = \dots = \phi_r = 0$ has infinitely many linearly independent solutions.*

Proof. Note first that the claim is obvious if $U \neq \emptyset$, because then $\text{Ker } \mathfrak{A}_{U,V}$ contains an infinite-dimensional space

$$\{u \mid u \in H^s(M), \text{ supp } u \subset U\}.$$

So it is sufficient to consider the case $U = \emptyset$, $V \neq \emptyset$. We will use the following simple inequality from linear algebra:

$$(1.7) \quad \dim(L_1 \cap L_2) \geq \dim L_2 - \text{codim } L_1,$$

where L_1, L_2 are subspaces in a linear space L , such that $\text{codim } L_1 < \infty$. Here $\text{codim } L_1 = \dim(L/L_1)$ is understood in a purely algebraic sense.

Now let us take $L = H_s(M, \partial M)$, $L_1 = \text{Im } \mathfrak{A}$ and

$$L_2 = \{(0, \phi_1, \dots, \phi_r) \mid \text{supp } \phi_j \subset V \text{ for all } j = 1, \dots, r\}.$$

Since \mathfrak{A} is Fredholm, we have $\text{codim } L_1 < \infty$. Also obviously $\dim L_2 = \infty$. Therefore by (1.7) we have $\dim L_1 \cap L_2 = \infty$. It remains to notice that the restriction of \mathfrak{A} maps surjectively $\text{Ker } \mathfrak{A}_{\emptyset,V}$ on $L_1 \cap L_2$, hence $\dim \text{Ker } \mathfrak{A}_{\emptyset,V} = \infty$. \square

Now we need to impose more restrictions on the problem (1.1). Recall the following

Definition 1.2. The problem (1.1) is called *normal* if the following conditions are satisfied:

- (i) A is properly elliptic;
- (ii) $m_i \neq m_j$ for all $i \neq j$;
- (iii) For every $j = 1, \dots, r$, the operator B_j contains highest order normal derivative with a non-vanishing coefficient at every point of the boundary (in other words the boundary is non-characteristic for B_j at all points).

Both the Dirichlet and Neumann problems are normal. The simplest example of an elliptic boundary value problem which is not normal is the following:

$$(T) \quad \begin{cases} \Delta u = f & \text{on } M, \\ \frac{\partial u}{\partial \tau} |_{\partial M} = \phi, \end{cases}$$

where M is a bounded connected domain with a connected C^∞ boundary in \mathbb{R}^2 , τ is a unit continuous tangent vector field on ∂M . The kernel of the corresponding operator again consists of constants only, and the problem is solvable if and only if

$$(1.8) \quad \int_{\partial M} \phi(s) ds = 0,$$

where ds means the element of the arc length.

Definition 1.3. Let A be a differential operator of order m on M . We will say that the *uniqueness of the Cauchy problem* holds for A if for any non-empty connected hypersurface $S \subset M$, not necessarily closed and possibly intersecting with the boundary (or even belonging to the boundary), the Cauchy problem for A with the Cauchy data on S can not have more than one solution in an open connected neighborhood of S in M .

More explicitly this means that the conditions

$$(1.9) \quad \begin{cases} Av = 0 \text{ on } U, \\ \frac{\partial^j v}{\partial \nu^j} = 0 \text{ on } S, \quad j = 0, \dots, m-1, \end{cases}$$

imply that $v \equiv 0$ on U . Here U is an open connected neighborhood of S , ν is a vector field of unit normals to S .

A weaker property is so called *unique continuation property*: if a solution v of $Av = 0$ on M vanishes in a non-empty open subset of M then $v \equiv 0$ on M .

The uniqueness of the Cauchy problem holds e.g. if A has real-analytic coefficients or if A is elliptic and has simple characteristics (real or complex), in particular if L is a second order elliptic operator with a real principal symbol (see e.g. [H], Sect. 17.2).

Theorem 1.4. *Let us consider a normal elliptic boundary value problem (1.1) such that the uniqueness of the Cauchy problem holds for the formally adjoint operator A^* . Then the “relaxed” elliptic boundary value problem (1.5) with one of the sets U, V non-empty is solvable for any f, ϕ_1, \dots, ϕ_r in appropriate spaces, i.e. the operator (1.6) is surjective.*

Here the formally adjoint operator is taken with respect to an inner product

$$(1.10) \quad (u, v)_M = \int_M u(x) \overline{v(x)} d\mu(x),$$

where $d\mu(x)$ is a chosen measure with a positive smooth density.

Remark 1.5. It is easy to see that neither normality condition nor the unique continuation property requirement can be omitted in Theorem 1.4. Indeed, if we consider the problem (T) above (which is not normal but satisfies the unique continuation property) and take “relaxed” problem (1.5) with $U \neq \emptyset$ and $V = \emptyset$, then the problem still will not be solvable unless the condition (1.8) is satisfied. On the other hand let us take an elliptic operator A with positive principal symbol such that A^* does not have unique continuation property (for examples see e.g. [Z]), so that there exists a solution $v \neq 0$ of $A^*v = 0$ with $\text{supp } v \subset \text{Int } M$. Consider the relaxed Dirichlet problem for this operator

$$(1.11) \quad \begin{cases} Au = f & \text{on } M \setminus U, \\ \frac{\partial^j u}{\partial \nu^j} |_{\partial M \setminus V} = \phi_j, & j = 0, \dots, m/2 - 1, \end{cases}$$

where $U \cap \text{supp } v = \emptyset$. The corresponding “non-relaxed” problem will be elliptic and normal. But the solvability of the problem (1.11) implies that $(f, v) = 0$, so it is not always solvable.

Before we embark on the proof of Theorem 1.4, let us discuss duality questions for normal elliptic boundary problems. We will mainly follow [A] and [LM], Sect.2 in Chapter 2, where one can find more details and references.

The system of boundary differential operators B_1, \dots, B_m of orders $\text{ord } B_j = m_j < m$ is called a *Dirichlet system* if it satisfies the requirements (ii) and (iii) of Definition 1.2, i.e. the set of all orders coincides with $\{1, 2, \dots, m\}$ and the boundary is non-characteristic for each B_j .

Any normal system $\{B_1, \dots, B_r\}$ (i.e. a system satisfying conditions (ii), (iii) of Definition 1.2) can be supplemented by operators B_{r+1}, \dots, B_m to a Dirichlet system $\{B_1, \dots, B_m\}$.

Let us fix a properly elliptic operator A on M and measures $d\mu, d\mu'$ with smooth positive densities on M and ∂M respectively. Then for any Dirichlet system $\{B_1, \dots, B_m\}$ there exists another Dirichlet system $\{C_1, \dots, C_m\}$ such that the following Green formula holds:

$$(1.12) \quad (Au, v)_M - (u, A^*v)_M = \sum_{j=r+1}^m (B_j u, C_{m+1-j} v)_{\partial M} - \sum_{j=1}^r (B_j u, C_{m+1-j} v)_{\partial M},$$

where the inner product $(\cdot, \cdot)_{\partial M}$ is defined as in (1.10) with M and $d\mu$ replaced by ∂M and $d\mu'$ respectively.

The boundary value problem

$$(1.13) \quad \begin{cases} A^*v = \tilde{f} & \text{on } M, \\ C_j v |_{\partial M} = \tilde{\phi}_j, & j = 1, \dots, r, \end{cases}$$

is called a *formally adjoint problem* to (1.1). The most important property of this problem is that the range of the operator \mathfrak{A} associated with the problem (1.1) (see (1.3)), is determined as the orthogonal complement (or rather annihilator) to the kernel of the formally adjoint problem (1.13). Namely,

(1.14)

$$\begin{aligned} \text{Im } \mathfrak{A} = \{ & (f, \phi_1, \dots, \phi_r) \in H_s(M, \partial M) \mid (f, v)_M + \sum_{j=1}^r (\phi_j, C_{m+1-j}v)_{\partial M} = 0 \\ & \text{for all } v \in C^\infty(M) \text{ such that } A^*v = 0 \text{ on } M \\ & \text{and } C_1v = \dots = C_rv = 0 \text{ on } \partial M \} \end{aligned}$$

(see [LM]). The formally adjoint problem to an elliptic boundary problem is also elliptic ([LM]). It follows that the set of v described in (1.14) is a finite-dimensional closed subspace in any Sobolev space.

Proof of Theorem 1.4. A. The proof splits into two cases: $U \neq \emptyset$ and $V \neq \emptyset$. Let us consider the case $U \neq \emptyset$ first. Then without loss of generality we can also assume that U has a smooth boundary (we can always take a smaller non-empty U) and also that $V = \emptyset$. Let us denote

$$N = \{(g, 0, \dots, 0) \mid g \in H^{s-m}(M), g = 0 \text{ on } M \setminus U\} \subset L = H_s(M, \partial M).$$

Solvability of the “relaxed” problem (1.5) for some fixed f, ϕ_1, \dots, ϕ_r is equivalent to

$$((f, \phi_1, \dots, \phi_r) + N) \cap \text{Im } \mathfrak{A} \neq \emptyset,$$

or, equivalently,

$$(f, \phi_1, \dots, \phi_r) \in N + \text{Im } \mathfrak{A}.$$

Since we want to prove this for arbitrary f, ϕ_1, \dots, ϕ_r , we need to prove that

$$(1.15) \quad N + \text{Im } \mathfrak{A} = L.$$

Note that $\text{Im } \mathfrak{A}$ is closed and has a finite codimension in L , which implies that the subspace $N + \text{Im } \mathfrak{A}$ is always closed in L . Let us consider its annihilator $(N + \text{Im } \mathfrak{A})^\circ$ in the dual space

$$L^* = H^{-s+m}(M) \times \prod_{j=1}^r H^{-s+m_j+1/2}(\partial M),$$

where the duality is taken with respect to the L^2 inner product

$$((u, \phi_1, \dots, \phi_r), (v, \psi_1, \dots, \psi_r)) = (u, v)_M + \sum_{j=1}^r (\phi_j, \psi_j)_{\partial M}.$$

Clearly this annihilator equals $N^\circ \cap (\text{Im } \mathfrak{A})^\circ$, so we need to prove

$$(1.16) \quad N^\circ \cap (\text{Im } \mathfrak{A})^\circ = \{0\}.$$

The description of the annihilator $(\text{Im } \mathfrak{A})^\circ$ follows from (1.14):

$$(1.17) \quad (\text{Im } \mathfrak{A})^\circ = \{(v, C_m v|_{\partial M}, \dots, C_{m-r+1} v|_{\partial M}) \mid v \in C^\infty(M), A^* v = 0 \text{ on } M, C_1 v|_{\partial M} = \dots = C_r v|_{\partial M} = 0.\}$$

On the other hand obviously

$$N^\circ = \{(g, \psi_1, \dots, \psi_r) \in L' \mid \text{supp } g \subset M \setminus U\}.$$

Now (1.16) becomes equivalent to the statement that the conditions

$$A^* v = 0 \text{ on } M, C_1 v|_{\partial M} = 0, \dots, C_r v|_{\partial M} = 0, \text{ and } v = 0 \text{ on } U$$

imply $v = 0$ on M . This immediately follows from the unique continuation property for A^* .

B. Now let us consider the case $U = \emptyset, V \neq \emptyset$. We will proceed similarly to the case when $U \neq \emptyset$. As in that case we can assume without loss of generality that V has a smooth boundary. Denote

$$N_1 = \{(0, \psi_1, \dots, \psi_r) \mid \psi_j \in H^{s-m_j-1/2}(\partial M), \psi_j = 0 \text{ on } \partial M \setminus V\} \subset L,$$

where $L = H_s(M, \partial M)$. The arguments given above show that we have to establish the equality

$$(1.18) \quad N_1^\circ \cap (\text{Im } \mathfrak{A})^\circ = \{0\}.$$

Clearly N_1° has the following description:

$$N_1^\circ = \{(g, \psi_1, \dots, \psi_r) \in L^* \mid \text{supp } \psi_j \subset \partial M \setminus V\}.$$

Using the description (1.17) for $(\text{Im } \mathfrak{A})^\circ$ we see that (1.18) reduces to the statement that the conditions

$$A^* v = 0 \text{ on } M, C_1 v|_{\partial M} = \dots = C_r v|_{\partial M} = 0, \text{ and } C_m v|_V = \dots = C_{m-r+1} v|_V = 0$$

imply $v = 0$ on M . Recalling that $r = m/2$ and $\{C_1, \dots, C_m\}$ is a Dirichlet system, we see that these conditions imply that the Cauchy data of v vanish on V i.e.

$$v = \frac{\partial v}{\partial \nu} = \dots = \frac{\partial^{m-1} v}{\partial \nu^{m-1}} = 0 \text{ on } V.$$

Due to the uniqueness of the Cauchy problem for A^* this implies $v = 0$ on M . \square

Remark. A.V.Fursikov pointed out that the relaxed problem (1.5) itself and the arguments given in the proof of Theorem 1.4 are similar to the ones used in control problems (see e.g. [L]). Indeed we actually force the fulfilling of the solvability conditions for the problem (1.1) by allowing “control” of the data f or ϕ_1, \dots, ϕ_r on “small” open sets U or V respectively.

2. ELLIPTIC BOUNDARY PROBLEMS ON COVERINGS

A. In this section we will describe how some of the (mostly classical) results of the previous section can be extended to a special but important non-compact situation: regular coverings of compact manifolds. This means that we will consider manifolds M with boundary and with a free C^∞ action of a discrete group Γ (we will say in this case that M is a Γ -manifold) such that the quotient space M/Γ is compact (hence M/Γ is a compact manifold with boundary, and we will say in this case that the action of Γ is *cocompact*). Note that the action of Γ maps ∂M into itself and $\partial M/\Gamma$ is a compact closed manifold. In this situation we will consider elliptic boundary value problems of the form (1.1) where all the coefficients of A and B_j are Γ -periodic. We will say in this case that the boundary value problem (1.1) is Γ -periodic.

At the same time we will consider this problem in uniform Sobolev spaces i.e. in the scales of spaces $H^s(M)$, $H^s(\partial M)$ which are based on L^2 spaces with respect to a smooth positive Γ -invariant density. On details about the definition and properties of these spaces see e.g. [R] or [Sh1] where a more general case of manifolds of bounded geometry is considered. These Sobolev spaces are Γ -invariant i.e. any element $\gamma \in \Gamma$ by the change of variables defines a unitary transformation γ^* on each of the Sobolev spaces.

Natural examples of Γ -periodic elliptic boundary value problems are Γ -invariant Dirichlet and Neumann problems (D) and (N) (Sect.1) on Γ -manifolds. In these problems instead of Δ we should take a Laplace–Beltrami operator of Γ -invariant Riemannian metric on M . If $\dim M = 2$, then we can also consider the problem (T). Here is the simplest example of this situation.

Example 2.1. Let M be a horizontal strip in \mathbb{R}^2 (e.g. $\{(x, y) \mid 0 \leq y \leq 1\}$) and $\Gamma = \mathbb{Z}$ acts by horizontal translations, i.e. $z \cdot (x, y) = (x + z, y)$, $z \in \mathbb{Z}$. Let Δ be the standard (flat) Laplacian in this strip. Then each of the problems (D), (N), (T) is a Γ -periodic elliptic boundary value problem.

As a result of application of general von Neumann algebra technique (based on M.Breuer theory of Fredholm operators in von Neumann algebras) we will see in particular that Proposition 1.1 extends to the Γ -invariant situation provided U and V are Γ -invariant.

B. Now we will describe necessary preliminaries on the von Neumann algebras technique. What we actually need is a quantitative characteristic of some Γ -invariant subspaces in the uniform Sobolev spaces – so called von Neumann Γ -dimension. This is a function which is defined on the set of all closed Γ -invariant subspaces, takes values in $[0, \infty]$, has many properties of the usual dimension, but at the same time takes finite values on many infinite-dimensional spaces.

We shall very briefly describe the definition and the necessary properties of the von Neumann Γ -dimension and Γ -trace. For more details we refer the reader to [At], [C] and textbooks on von Neumann algebras (e.g. [D], [N], [T]).

We shall denote the Γ -dimension by \dim_Γ . It is defined on the set of all (projective) Hilbert Γ -modules and takes values in $[0, \infty]$. The simplest Hilbert Γ -module is given by a left regular representation of Γ : it is the Hilbert space $L^2\Gamma$ consisting

of all complex-valued L^2 -functions on Γ . The group Γ acts unitarily on $L^2\Gamma$ by $\gamma \mapsto L_\gamma$ where L_γ is defined as follows:

$$L_\gamma f(x) = f(\gamma^{-1}x), \quad x \in \Gamma; \quad f \in L^2\Gamma.$$

By definition $\dim_\Gamma L^2\Gamma = 1$.

For any (complex) Hilbert space \mathcal{H} define a free Hilbert Γ -module $L^2\Gamma \otimes \mathcal{H}$. Its Γ -dimension equals $\dim_{\mathbb{C}} \mathcal{H}$. The action of Γ in $L^2\Gamma \otimes \mathcal{H}$ is defined by $\gamma \mapsto L_\gamma \otimes I$.

A general Hilbert Γ -module is a closed Γ -invariant subspace in a free Hilbert Γ -module. It would be natural to call such subspaces *projective* Hilbert modules, but the word “projective” is usually omitted, so only projective Hilbert modules are considered.

For any Hilbert space \mathcal{H} denote by \mathcal{A}_Γ a von Neumann algebra which consists of all bounded linear operators in $L^2\Gamma \otimes \mathcal{H}$ which commute with the action of Γ there. This algebra is in fact generated by the operators of the form $R_\gamma \otimes B$, $B \in \mathcal{B}(H)$, $\gamma \in \Gamma$, where $\mathcal{B}(H)$ is the algebra of all bounded linear operators in \mathcal{H} , R_γ is the operator of the right translation in $L^2\Gamma$ i.e.

$$R_\gamma f(x) = f(x\gamma), \quad x \in \Gamma; \quad f \in L^2\Gamma.$$

This means that the algebra \mathcal{A}_Γ is the weak closure of all finite linear combinations of the operators of the form $R_\gamma \otimes B$. So in fact \mathcal{A}_Γ is a tensor product (in the sense of von Neumann algebras) of \mathcal{R}_Γ and $\mathcal{B}(H)$ where \mathcal{R}_Γ is the von Neumann algebra generated by the operators R_γ in $L^2\Gamma$ (it consists of all operators in $L^2\Gamma$ which commute with all operators L_γ , $\gamma \in \Gamma$).

There is a natural trace on \mathcal{R}_Γ . It is denoted by tr_Γ and defined as the diagonal matrix element (all of them are equal) in the δ -functions basis. For example we can define it by

$$\text{tr}_\Gamma S = (S\delta_e, \delta_e), \quad S \in \mathcal{R}_\Gamma,$$

where e is the neutral element of Γ , $\delta_e \in L^2\Gamma$ is the “Dirac delta-function” at e , i.e. $\delta_e(x) = 1$ if $x = e$ and 0 otherwise. There is a natural trace on \mathcal{A}_Γ too: $\text{Tr}_\Gamma = \text{tr}_\Gamma \otimes \text{Tr}$ where Tr is the usual trace on $\mathcal{B}(H)$.

Now for any Hilbert Γ -module which is a Γ -invariant subspace L in $L^2\Gamma \otimes \mathcal{H}$, its Γ -dimension is defined by the natural formula

$$\dim_\Gamma L = \text{Tr}_\Gamma P_L,$$

where P_L is the orthogonal projection on L in $L^2\Gamma \otimes \mathcal{H}$.

We will only need the Γ -modules which appear as closed Γ -invariant subspaces of the (free) Hilbert Γ -module $L^2(M)$, where the space $L^2(M)$ is defined by a measure with a smooth positive Γ -invariant density (e.g. volume element associated with the chosen Γ -invariant riemannian metric on M). Let F be a fundamental domain of the action of Γ . Suppose that an operator A in $L^2(M)$ has a finite Γ -trace and a continuous Schwartz kernel $K_A(x, y)$, so

$$Au(x) = \int_M K_A(x, y)u(y)d\mu(y), \quad u \in L^2(M),$$

where $d\mu$ is the (Γ -invariant) measure. Then

$$\mathrm{Tr}_\Gamma A = \int_F K_A(x, x) d\mu(x)$$

(see e.g. [At]).

Similar definitions and formulas apply to the subspaces and operators on the boundary ∂M of a Γ -manifold with boundary M .

C. Let us recall necessary elements of Breuer Fredholm theory [B] in the necessary context (see [Sh2] for more details).

Let $A : L_1 \rightarrow L_2$ be a bounded linear operator, where L_1, L_2 are Hilbert Γ -modules. Then A is called *morphism of Hilbert Γ -modules* if it commutes with the action of Γ , i.e. if it is an intertwining operator between two canonical representations of Γ in L_1 and L_2 .

Definition 2.2. A morphism of Hilbert Γ -modules $A : L_1 \rightarrow L_2$ is called a Γ -Fredholm operator if the following two conditions are satisfied

- (i) $\dim_\Gamma \mathrm{Ker} A < \infty$;
- (ii) there exists a closed Γ -invariant subspace $\tilde{L} \subset L_2$, such that $\tilde{L} \subset \mathrm{Im} A$ and $\dim_\Gamma(L_2 \ominus \tilde{L}) < \infty$.

It is easy to see that for the case of the trivial group $\Gamma = \{e\}$ this definition gives us the usual Fredholm operators.

As in the case of trivial Γ , the set of all Γ -Fredholm operators is an open subset (in norm-topology) in the set of all morphisms $A : L_1 \rightarrow L_2$ of Hilbert Γ -modules.

For infinite Γ the image $\mathrm{Im} A$ of a Γ -Fredholm operator is not necessarily closed but it is *almost closed* i.e. for every $\varepsilon > 0$ there exists a closed Γ -invariant subspace $\tilde{L}_\varepsilon \subset L_2$ such that $\tilde{L}_\varepsilon \subset \mathrm{Im} A$ and $\dim_\Gamma(\overline{\mathrm{Im} A} \ominus \tilde{L}_\varepsilon) < \varepsilon$. Note that for the trivial Γ this would mean that $\mathrm{Im} A$ is closed because the usual dimension can not take values in $(0, 1)$.

Let L be a Hilbert Γ -module, \mathcal{A}_Γ the von Neumann algebra associated with L . Let us recall definitions of some important ideals in \mathcal{A} .

The ideal of all Γ -trace class operators is the set of all finite linear combinations of operators $A \in \mathcal{A}_\Gamma$ such that $A \geq 0$ and $\mathrm{Tr}_\Gamma A < \infty$. Denote this set by $J_1(\mathcal{A}_\Gamma)$. It is a two-sided ideal in \mathcal{A}_Γ . The Γ -trace can be uniquely extended to a linear map $\mathrm{Tr}_\Gamma : J_1(\mathcal{A}_\Gamma) \rightarrow \mathbb{C}$.

An operator $B \in \mathcal{A}_\Gamma$ is called a Γ -Hilbert-Schmidt operator if $B^*B \in J_1(\mathcal{A}_\Gamma)$ (or $\mathrm{Tr}_\Gamma(B^*B) < \infty$). Denote $J_2(\mathcal{A}_\Gamma)$ the set of all Γ -Hilbert-Schmidt operators. It is also a two-sided ideal in \mathcal{A}_Γ and $(J_2(\mathcal{A}_\Gamma))^2 = J_1(\mathcal{A}_\Gamma)$.

The operators in the norm closure of $J_1(\mathcal{A}_\Gamma)$ (or $J_2(\mathcal{A}_\Gamma)$) are called Γ -compact. They also form a two-sided ideal in \mathcal{A}_Γ which is denoted $J_\infty(\mathcal{A}_\Gamma)$.

It can be proved (see [B]) that a morphism $A : L_1 \rightarrow L_2$ of Hilbert Γ -modules is Γ -Fredholm if and only if it is invertible modulo J_1 , J_2 or J_∞ . For example to establish that A is Γ -Fredholm it is sufficient to find two morphisms $B_1, B_2 :$

$L_2 \rightarrow L_1$ of Hilbert Γ -modules such that both $B_1A - \text{Id}_{L_1}$ and $AB_2 - \text{Id}_{L_2}$ are Γ -Hilbert-Schmidt operators in L_1 and L_2 respectively.

We will need a condition for an operator to be Γ -Hilbert-Schmidt in the geometric situation described above: in the Hilbert Γ -module $L^2(M)$. Note that a bounded linear operator A in $L^2(M)$ commutes with the action of Γ on M if and only if for the Schwartz kernel K_A of A we have

$$K_A(\gamma x, \gamma y) = K_A(x, y), \quad x, y \in M, \quad \gamma \in \Gamma .$$

Therefore in this case K_A is a distribution on $(M \times M)/\Gamma$ where Γ acts diagonally on $M \times M$. Let us consider the product measure on $M \times M$ and the induced measure on the orbit space $(M \times M)/\Gamma$.

The following observation is implicit in [At]:

Lemma 2.4. *Let A be an endomorphism of the Hilbert Γ -module $L^2(M)$, K_A its Schwartz kernel. Then A is Γ -Hilbert-Schmidt if and only if $K_A \in L^2((M \times M)/\Gamma)$.*

Now we can formulate a Γ -analogue of the main property of elliptic boundary problems on compact manifolds.

Theorem 2.5. *Consider a Γ -invariant elliptic boundary problem (1.1). Then the corresponding operator (1.3) in uniform Sobolev spaces is Γ -Fredholm.*

Sketch of the proof. Using local parametrices of the boundary problem (1.1) (see e.g. [Se], [RS] or [Gr]) and an appropriate Γ -invariant partition of unity, we can construct an ‘‘almost local’’ Γ -invariant parametrix $\mathfrak{P} : H_s(M, \partial M) \rightarrow H^s(M)$ of the operator \mathfrak{A} in (1.3) so that the Schwartz kernel of \mathfrak{P} is supported near the diagonal, and both $\mathfrak{P}\mathfrak{A} - \text{Id}$ and $\mathfrak{A}\mathfrak{P} - \text{Id}$ have smooth kernels, also supported near the diagonal. It is easy to deduce then from Lemma 2.4 that the operators $\mathfrak{P}\mathfrak{A} - \text{Id}$ and $\mathfrak{A}\mathfrak{P} - \text{Id}$ will be Γ -Hilbert-Schmidt. It follows that \mathfrak{A} is Γ -Fredholm. \square

Now let us consider an elliptic Γ -invariant problem (1.1) and the corresponding ‘‘relaxed’’ problem (1.5) where U, V are open and Γ -invariant, one of them non-empty. We will only prove an analogue of Proposition 1.1.

Proposition 2.6. *For the operator $\mathfrak{A}_{U,V}$ corresponding to a relaxed Γ -invariant problem (1.5) and defined by (1.6), we have $\dim_{\Gamma} \text{Ker } \mathfrak{A}_{U,V} = \infty$.*

Proof. Note that the following analogue of (1.7) holds for the Γ -dimension:

$$\dim_{\Gamma}(L_1 \cap L_2) \geq \dim_{\Gamma} L_2 - \text{codim}_{\Gamma} L_1 ,$$

where L_1, L_2 are closed Γ -invariant subspaces of a Hilbert Γ -module L (see [GHS]). It remains to repeat the proof of Proposition 1.1 verbatim. \square

Corollary 2.7. *Under the conditions of Proposition 2.6, $\dim \text{Ker } \mathfrak{A}_{U,V} = \infty$.*

Proof. Note that $\dim L \geq \dim_{\Gamma} L$ for any Hilbert Γ -module L . \square

3. OPEN PROBLEMS

The following open problems naturally arise from the context above and we plan to return to them in the future.

Problem 1. Consider the following “relaxed” problem to (1.1), more general than (1.5):

$$(3.1) \quad \begin{cases} Au = f & \text{on } M \setminus U, \\ B_j u|_{\partial M \setminus V_j} = \phi_j, & j = 1, \dots, r, \end{cases}$$

where V_1, \dots, V_r are non-empty open subsets in ∂M (possibly disjoint).

- A. Does the solvability result (Theorem 1.4) hold for this problem?
- B. Will the solvability hold if we allow some of the sets V_1, \dots, V_r to be empty?

The difficulty here is that the dual statement to solvability does not reduce to the uniqueness of the Cauchy problem. However sometimes the corresponding uniqueness result might still hold. The following example of this kind is due to N.Nadirashvili.

Let $M = B$ be the unit ball $\{x \mid |x| \leq 1\}$ in \mathbb{R}^n , Δ is the flat Laplacian on \mathbb{R}^n . Consider the following boundary problem:

$$(3.1) \quad \begin{cases} \Delta^2 u = 0 \text{ in } B, \\ u = 0 \text{ on } \partial B, \\ \frac{\partial u}{\partial \nu} = \frac{\partial^2 u}{\partial \nu^2} = 0 \text{ on } V, \end{cases}$$

where V is a non-empty open piece of the sphere ∂B . It follows that $u \equiv 0$. Indeed, u can be presented in the form

$$u(x) = v(x) + (1 - |x|^2)w(x), \quad \Delta v = \Delta w = 0.$$

(See e.g. [TS]). It follows immediately from (3.1) that $v = 0$ on ∂B , hence $v \equiv 0$. Now the remaining boundary conditions in (3.1) imply $w = \frac{\partial w}{\partial \nu} = 0$ on V , hence $w \equiv 0$ due to the uniqueness of the Cauchy problem for the equation $\Delta w = 0$. So we proved that the conditions (3.1) imply $u \equiv 0$.

Problem 2. Consider a relaxed problem (1.5) on a covering manifold (as in Section 2). Does the solvability result hold in this situation?

Here the difficulty is that the duality argument can not be applied directly because the image of the corresponding Γ -Fredholm operator can be not closed.

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