

INCREASING ASYMPTOTIC SOLUTIONS OF THE KORTEWEG-DE VRIES EQUATION AND ITS HIGHER ANALOGUES

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We consider the Cauchy problem for the Korteweg-de Vries (KdV) equation

$$(1) \quad u_t - 6uu_x + u_{xxx} = 0,$$

$$(2) \quad u|_{t=0} = u_0(x).$$

An extensive literature is devoted to the KdV equation and other equations integrable by the method of the inverse problem of scattering theory and its modifications (see, for example, [1] and the bibliography there). In this literature decreasing or periodic (in x) solutions are mainly considered, although there are some results concerning increasing solutions (see, for example, [2] or [3]).

In this note we show that if in problem (1), (2) (or in the Cauchy problem for one of the higher KdV equations) the initial function $u_0(x)$ has an asymptotic expansion in increasing powers of x , understood in the same sense as asymptotic expansions of symbols in the theory of pseudodifferential operators, then it is possible to find an asymptotic solution of this problem explicitly (up to functions of the Schwartz class $S(\mathbb{R})$ in x). This solution for the l th KdV equation is defined for all t if and only if $u_0(x) = o(|x|^{1/(l-1)})$ as $|x| \rightarrow \infty$ (equation (1) corresponds to the case $l = 2$). If o is replaced by O the solution already becomes defined only on a finite interval in t , while if $u_0(x)$ contains in its asymptotics (as $x \rightarrow +\infty$ or $-\infty$) a term $|x|^{(1+\varepsilon)/(l-1)}$, with $\varepsilon > 0$, then there exists no solution belonging to the class considered on any interval (or even half-open interval) of the t -axis containing 0.

1. We recall the formulas defining the l th KdV equation in the formalism of Gel'fand and Dikii [4]. This equation has the form

$$(3_l) \quad u_t = (\partial/\partial x)R_l(u),$$

where $R_l(u)$ are polynomials in u, u', u'', \dots (here $u' = u_x, u'' = u_{xx}$, etc.) defined by the recurrence formulas

$$R_{l+1} = -2 \sum_{k=0}^l R_k R_{l-k}' + \sum_{k=0}^l R_k' R_{l-k}' + 4u \sum_{k=0}^l R_k R_{l-k} - \sum_{k=1}^l R_k R_{l-k+1}$$

and the initial conditions

$$R_0 = 1/2, \quad R_1 = u.$$

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In particular, it follows that $R_2 = 3u^2 - u''$, and (3₂) has the form (1). Moreover, the functions R_l satisfy the simpler recurrence relation

$$(4) \quad R'_{l+1} = -R'''_l + 4uR'_l + 2u'R_l.$$

We now introduce the class F_α^+ of formal power series of the form

$$(5) \quad u = \sum_{j=0}^{\infty} c_j x^{\alpha_j},$$

where $c_j \in \mathbb{C}$, $\alpha_j \in \mathbb{R}$, $\alpha_0 > \alpha_1 > \alpha_2 > \dots$, $\alpha_j \rightarrow -\infty$ as $j \rightarrow +\infty$ and $\alpha_0 = \alpha$. Below series of the form (5) give the asymptotics of genuine functions $u(x)$ as $x \rightarrow +\infty$. The asymptotics as $x \rightarrow -\infty$ are given by analogous series with x replaced by $-x$. We denote the corresponding class of series by F_α^- . It is convenient to describe both asymptotics as $x \rightarrow \pm\infty$ by a pair of series, i.e., an element of the space $F_\alpha = F_\alpha^+ \oplus F_\alpha^-$.

Further, for any interval $I \subset \mathbb{R}$ (open or closed) let $C^\infty(I, F_\alpha^+)$ denote the set of series of the form (5) where $c_j = c_j(t) \in C^\infty(I)$. The meaning of $C^\infty(I, F_\alpha^-)$ and $C^\infty(I, F_\alpha)$ is similar. In all these classes of formal series there are natural operations of differentiation (with respect to t and x) and multiplication. Substitution of series of this form into (3_l) is therefore meaningful. To simplify notation we henceforth consider only asymptotics as $x \rightarrow +\infty$, and we shall also assume that $\alpha > 0$. The following result is proved with the help of (4) by induction on l .

LEMMA 1. *If $u \in F_\alpha^+$ or $u \in C^\infty(I, F_\alpha^+)$, then $R_l(u) \in F_{l\alpha}^+$ or $R_l(u) \in C^\infty(I, F_{l\alpha}^+)$ respectively. The leading coefficient $r_{0,l}$ (of the power $x^{l\alpha}$) in the expansion of $R_l(u)$ can hereby be expressed in terms of the leading coefficient c_0 (of x^α) in the expansion of u by the formula*

$$r_{0,l} = A_l c_0^l, \quad A_l = (l!)^{-1} \cdot 2^{l-1} (2l-1)!!.$$

By virtue of this lemma the leading coefficient on the right side of (3_l) for $u \in C^\infty(I, F_\alpha^+)$ has the form $A_l l \alpha c_0^l x^{l\alpha-1}$. Therefore, if u satisfies (3_l) on a nontrivial interval I and $c_0(t) \not\equiv 0$ on I , then we must have $l\alpha - 1 \leq \alpha$, i.e., $\alpha \leq (l-1)^{-1}$. In particular, if a solution of the Cauchy problem for (3_l) with initial condition $u_0 \in F_\alpha^+$ exists in any one of the classes $C^\infty(I, F_\beta^+)$ (here $0 \in I$) and if $u_0 \notin F_\alpha^+$ for any $\alpha' < \alpha$ (i.e., the coefficients of x^α in the expansion of u_0 is nonzero), then $\alpha \leq (l-1)^{-1}$. Thus, a solution $u \in F_\alpha^+$ with leading coefficient $c_0(t) \not\equiv 0$ (which can always be assumed with no loss of generality) can exist only for $\alpha \leq (l-1)^{-1}$. In the case $\alpha = (l-1)^{-1}$ we obtain for $c_0(t)$ the equation $\dot{c}_0 = B_l c_0^l$, where $B = A_l l \alpha$. It has a solution defined on the ray $(-\infty, t_0)$ or $(t_0, +\infty)$ depending on the sign of the initial condition $c_0(0)$. In the case $\alpha < (l-1)^{-1}$ we obtain for c_0 the equation $\dot{c}_0 = 0$, whence $c_0(t) = c_0(0) = \text{const}$.

2. We shall now consider the description of the remaining coefficients in the expansion of the formal solution.

LEMMA 2. *Let $A = \{\alpha_j\}_0^\infty$ be the collection of all the exponents in the expansion of $u \in C^\infty(I, F_\alpha^+)$, where $\alpha \leq (l-1)^{-1}$. Then the set of exponents in the expansion of $(\partial/\partial x)R_l(u)$ (the right side of (3_l)) is contained in the set \bar{A} of all numbers of the form*

$$(6) \quad \bar{\alpha} = \sum_{p=1}^k \alpha_{i_p} - 2(l-k) - 1, \quad k = 1, 2, \dots, l,$$

where i_1, \dots, i_k are any nonnegative integers (not necessarily distinct).

The proof follows easily by induction on l with the help of (4).

To ascribe a meaning to equation (3_l) with initial conditions (2) where $u_0 \in F_\alpha^+$ it is now necessary to extend the collection $A_0 = \{\alpha_j\}_0^\infty$ consisting of the exponents in the expansion of u_0 to a minimal collection A which does not change on adjoining all numbers $\bar{\alpha}$ of the form (6) constructed on the basis of any numbers $\alpha_{i_1}, \dots, \alpha_{i_k}$ of the collection A , i.e., to a collection A such that $\bar{A} \subset A$. It is clear that the corresponding minimal set A exists and is unique; however, it is not immediately evident that it is discrete and upper semibounded.

LEMMA 3. *Let $A_0 = \{\alpha_j\}_0^\infty$ be a discrete collection of numbers $\alpha_j \in \mathbb{R}$ such that $\alpha = \sup_j \alpha_j \leq (l-1)^{-1}$, and let A be the minimal set formed on the basis of it which contains A_0 and satisfies the condition $\bar{A} \subset A$. Then A is discrete and $\sup_{\beta \in A} \beta \leq (l-1)^{-1}$.*

For the proof we observe that we can adjoin to A_0 the number -2 , and in place of (6) we can write

$$\bar{\alpha} = \sum_{p=1}^l \alpha_{i_p} - 1.$$

Setting $\beta_j = \alpha_j - (l-1)^{-1}$ and $\bar{\beta} = \bar{\alpha} - (l-1)^{-1}$, we obtain $\bar{\beta} = \sum_{p=1}^l \beta_{i_p}$, whence it is evident that the set $B = \{\bar{\alpha} - (l-1)^{-1} : \bar{\alpha} \in A\}$ is contained in the set $\bar{B} = \{\sum_{p=1}^l n_p \beta_{i_p} : n_p \in \mathbb{Z}, n_p \geq 0\}$, i.e., \bar{B} is the semigroup generated by the set $\{\beta_j\}_0^\infty$. Since $\beta_j \leq 0$ and the set $\{\beta_j\}_0^\infty$ is discrete, \bar{B} is also discrete and $\bar{B} \subset (-\infty, 0]$; all the assertions of Lemma 3 follow from this. We note that $\bar{\beta} \leq \beta_{i_p}$, $p = 1, \dots, l$, and equality is possible only if $l-1$ of the numbers β_{i_p} are equal to 0. This implies that $\bar{\alpha} \leq \alpha_{i_p}$, $p = 1, \dots, l$, and the equality $\bar{\alpha} = \alpha_{i_p}$ is possible only under the condition that $l-1$ of the numbers α_{i_p} are equal to $(l-1)^{-1}$. From this it follows that equation (3_l) for $u \in C^\infty(I, F_\alpha^+)$ of the form (5) and for a collection $\{\alpha_j\}_0^\infty = A$ goes over into an equation with coefficients c_j , $j = 1, 2, \dots$, having the following form:

$$\dot{c}_j = \begin{cases} P_j(c_0, c_1, \dots, c_{j-1}), & \text{if } \alpha < (l-1)^{-1}, \\ E_j c_0^{-1} c_j + Q_j(c_0, c_1, \dots, c_{j-1}), & \text{if } \alpha = (l-1)^{-1}, \end{cases}$$

where the P_j and Q_j are polynomials and the E_j are constants. Applying induction on j , we see that if $\alpha < (l-1)^{-1}$, then the $c_j(t)$ are polynomials in t , while if $\alpha = (l-1)^{-1}$, then $c_j \in C^\infty(I)$, where I is that interval on which $c_0(t)$ is defined. In summary we obtain the following result.

THEOREM 1. a) *If $\alpha < (l-1)^{-1}$, then problem (3_l), (2) with initial condition $u_0 \in F_\alpha^+$ has a unique solution $u \in C^\infty(\mathbb{R}, F_\alpha^+)$, and in expansion (5) for this solution $c_0 = \text{const}$ and the $c_j(t)$ are polynomials in t for any $j = 1, 2, \dots$.*

b) *If $\alpha = (l-1)^{-1}$, then problem (3_l), (2) with initial condition $u_0 \in F_\alpha^+$ has a unique solution $u \in C^\infty(I, F_\alpha^+)$, where $I = (-\infty, t_0)$, $t_0 > 0$, or $I = (t_0, +\infty)$, $t_0 < 0$. If here $u_0 \notin F_\alpha^+$ with $\alpha' < (l-1)^{-1}$, then no solution in $C^\infty(\mathbb{R}, F_\beta^+)$ exists for any β .*

c) *If $\alpha > (l-1)^{-1}$ and $u_0 \in F_\alpha^+ \setminus F_{(l-1)^{-1}}^+$, then problem (3_l), (2) has no solution $u \in C^\infty(I, F_\beta^+)$ for any β and any interval I containing 0.*

The assertions of the theorem remain true with the classes F_α^+ replaced by F_α^- and F_α .

3. We say that a function $u(x, t) \in C^\infty(\mathbf{R} \times I)$ has asymptotics $\bar{u} \in C^\infty(I, F_\alpha)$ as $|x| \rightarrow \infty$, where

$$\bar{u}(x, t) = \left(\sum_{j=0}^{\infty} c_j^+(t) x^{\alpha_j}, \sum_{j=0}^{\infty} c_j^-(t) (-x)^{\alpha_j} \right),$$

if

$$\left| \partial_t^k \partial_x^m \left[u(\pm x, t) - \sum_{j=0}^N c_j^\pm(t) (\pm x)^{\alpha_j} \right] \right| \leq C_{k,m,N,J} |x|^{\alpha_{N+1}-m}, \quad t \in J, x \geq 1,$$

where J is any bounded interval of \mathbf{R} such that $\bar{J} \subset I$; k, m , and N are any nonnegative integers. In this case we write $u \sim \bar{u}$. If $u_0 \in C^\infty(\mathbf{R})$ and $\bar{u}_0 \in F_\alpha$, the meaning of the expression $u_0 \sim \bar{u}_0$ is similar. A standard argument of the theory of pseudodifferential operators (see, for example, [5], §3) shows that for any formal series $\bar{u}_0 \in F_\alpha$ there exists a function $u_0 \in C^\infty(\mathbf{R})$ with asymptotics \bar{u}_0 . It is proved similarly that for any $\bar{u} \in C^\infty(I, F_\alpha)$ there exists a function $u \in C^\infty(\mathbf{R} \times I)$ such that $u \sim \bar{u}$. This remark makes it possible to pass from formal solutions to asymptotic solutions. Namely, we call a function $u(x, t) \in C^\infty(\mathbf{R} \times I)$ an asymptotic solution of the Cauchy problem (3_l), (2) if

$$u_t - (\partial/\partial x)R_l(u) \in C^\infty(I, S(\mathbf{R})), \quad u|_{t=0} - u_0(x) \in S(\mathbf{R}),$$

where $C^\infty(I, S(\mathbf{R}))$ is the space of smooth functions on I with values in $S(\mathbf{R})$ understood in the obvious sense. From Theorem 1 we obtain the following result.

THEOREM 2. *If the initial datum u_0 has asymptotics $\bar{u}_0 \in F_\alpha$, where $\alpha \leq (l-1)^{-1}$, then the Cauchy problem (3_l), (2) has an asymptotic solution $u(x, t) \in C^\infty(\mathbf{R} \times I)$, where I is the same sort of interval as in Theorem 1 (in particular, for $(l-1)^{-1}$ it is possible to take $I = \mathbf{R}$). The asymptotic expression \bar{u} of the solution $u(x, t)$ is a formal solution of (3_l) with initial condition $\bar{u}|_{t=0} = \bar{u}_0$. If $\bar{u}_0 \in F_\alpha \setminus F_{(l-1)^{-1}}$, then problem (3_l), (2) has no solution with asymptotics in any class $C^\infty(I, F_\beta)$.*

For the KdV equation (i.e., in the case $l=2$) the first author has proved that if $\alpha < (l-1)^{-1}$, then in the situation of Theorem 2 there also exists a genuine solution of problem (1), (2) differing from the asymptotic solution by a function in $C^\infty(\mathbf{R}, S(\mathbf{R}))$.

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