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# UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM FOR THE KORTEWEG - DE VRIES EQUATION IN CLASSES OF INCREASING FUNCTIONS

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The authors prove the uniqueness of the solution of the Cauchy problem for the Korteweg - de Vries equation on $[0, T]\times\mathbf{R}_x$ in the class of functions u(t, x) that satisfy the following conditions as $|x| \to \infty$ uniformly with respect to $t\in[0, T]$:

$$u(t, x)=o(|x|);\quad \frac{\partial^n u}{\partial x^n}=O(1), \qquad n=1, 2, 3, \ldots$$

Menikoff [1] proved the existence of a solution for the Cauchy problem for the Korteweg - de Vries (KdV) equation under an initial condition from some class of increasing functions. In this paper, we will establish that the solution is unique in a somewhat broader class of functions. In [1], uniqueness of the solution was proved only in the class of functions that have a certain asymptotic behavior as $|x| \to \infty$. In classes such as Menikoff classes [1], it is also possible to prove that a solution exists (this was done in [2] for somewhat narrower classes). Thus, in these classes a solution exists and is unique.

The basic idea of this paper is to utilize a variant of Holmgren's principle, applied to a linear equation in which the coefficients are expressed in terms of unknown functions. The existence of a solution of the conjugate equation (in the class of decreasing functions) is proved on the basis of the same ideas as in [2], which offered a solution for the equation that specifies the discrepancy between the asymptotic solutions of [3] and the actual solutions.

1. Consider the Cauchy problem for the KdV equation:

$$u_t+uu_x+u_{xxx}=0, \quad u|_{t=0}=u_0(x). \tag{1}$$

We will consider only smooth solutions $u=u(t, x)\in C^\infty([0, T]\times R^1)$. We set $u^{(n)}=\partial^n u/\partial x^n$. In [1], under the following conditions:

$$u_0^{(n)}(x)=o(|x|^{1-n}) \text{ for } |x|\to\infty, \quad n=0, 1, 2, \ldots, \tag{2}$$

it was shown that there exists a smooth solution of problem [1], defined for all t and x. The solution obtained in [1] satisfies the conditions

$$u^{(n)}(t, x)=o(|x|^{1-n}) \text{ for } |x|\to+\infty \text{ uniformly with respect to } t\in[-T, T], \tag{3}$$

where n = 0, 1, 2, ... and T > 0.

The following theorem implies, in particular, that the solution of problem (1) is unique in class (3). Thus, the class of initial conditions that satisfy (2) is invariant relative to the flux specified by the KdV equation.

**Theorem 1.** The solution of problem (1) is unique in the class of functions $u\in C^\infty([0, T]\times R^1)$, such that as $|x| \to \infty$, uniformly with respect to $t\in[0, T]$

$$u(t, x)=o(|x|); \quad u^{(n)}(t, x)=O(1), \quad n=1, 2, 3, \ldots \tag{4}$$

Of course, instead of the segment [0, T] we can employ [-T, 0].

**Remark.** Menikoff [1] proved the uniqueness of a solution u of problem (1) which differs from the solution v of the problem

$$v_t+vv_x=0, \quad v|_{t=0}=u_0(x)$$

by virtue of a function that decreases rapidly enough for the energy inequalities to be employed.

2. Let us begin the proof of Theorem 1. For simplicity, we will confine ourselves to real solutions, and we will assume that all the functions under consideration are real-valued. Assume that u and v are two solutions of problem (1) that satisfy condition (4). Then the function w = u - v also satisfies condition (4) and is a solution of the problem

$$w_t+uw_x+v_xw+w_{xxx}=0, \quad w|_{t=0}=0. \tag{5}$$

We wish to show that $w(t, x) \equiv 0$ for $t\in[0, T]$, $x\in R$. For this it is sufficient to establish that for arbitrary $t_0 \in [0, T]$ and $z_0 = z_0(x) \in C_0^\infty(\mathbf{R})$

$$(w(t_0, \cdot), z_0(\cdot)) = \int_{-\infty}^{\infty} w(t_0, x) z_0(x)\, dx = 0 \qquad (6)$$

(here and henceforth, $(\cdot, \cdot)$ and $\|\cdot\|$ denote the scalar product in space $L^2(\mathbf{R}^1_x)$ and the corresponding norm). Assume that we are given function $z = z(t, x) \in C^\infty([0, t_0], S(\mathbf{R}_x))$ (here $S(\mathbf{R}_x)$ is an ordinary Schwartz space of functions of the variable $x \in \mathbf{R}$). Integrating by parts, we obtain

$$\frac{d}{dt}(w(t, \cdot), z(t, \cdot)) = (w_t, z) + (w, z_t) =$$
$$= (-uw_x - v_x w - w_{xxx}, z) + (w, z_t) = (w, (uz)_x - v_x z + z_{xxx} + z_t).$$

Now, if we assume that z is the solution of the problem

$$z_t + (uz)_x - v_x z + z_{xxx} = 0, \qquad 0 < t < t_0; \quad z|_{t=t_0} = z_0(x), \qquad (7)$$

then we obtain

$$(w(t, \cdot), z(t, \cdot)) = \text{const},$$

from which we have

$$(w(t_0, \cdot), z_0(\cdot)) = (w(0, \cdot), z(0, \cdot)) = 0$$

in view of the initial condition for w (see (5)). Thus, expression (6) is observed in this case. Therefore, to prove Theorem 1 it is sufficient to establish that problem (7) is solvable in the class of functions $z \in C^\infty([0, t_0], S(\mathbf{R}_x))$ for any initial given $z_0 \in C_0^\infty(\mathbf{R})$ and for any $t_0 \in [0, T]$.

We can replace t by $t_0 - t$ in (7). This means that in (7) we must add a minus sign in front of $z_t$; the initial data will be specified for $t = 0$, while the problem should be solved for $t \in [0, t_0]$ in the same class of functions z. In exactly the same way, we can replace x by $-x$, and this causes the signs in front of $z_x$ and $z_{xxx}$ to change.

Now the following more general theorem implies that problem (7) is solvable.

Theorem 2. The problem

$$z_t + az + bz_x + z_{xxx} = 0, \quad z|_{t=0} = z_0(x) \tag{8}$$

has a solution $z_0 \in C^\infty([0, T], S(\mathbf{R}_x))$ for any initial condition $z_0 \in S(\mathbf{R})$, if the coefficients $a, b \in C^\infty([0, T] \times \mathbf{R})$ satisfy the following conditions as $|x| \to \infty$ uniformly with respect to $t \in [0, T]$:

$$a^{(n)}(t, x) = O(1), \qquad n = 0, 1, 2, \ldots; \tag{9}$$

$$b(t, x) = o(|x|); \quad b^{(n)}(t, x) = O(1), \quad n = 1, 2, 3, \ldots \tag{10}$$

3. Theorem 2 can be proved in the same way used for the proof of solvability of the Cauchy problem for the analogous equation in [2] (which is even more complicated, in that it contains a nonlinear term; see Eq. (2) in [2]). We will describe the principal steps of this proof.

To simplify the notation, we will assume that functions $a(t, x)$ and $b(t, x)$ are defined for all $t \in \mathbf{R}$ and satisfy conditions (9) and (10) uniformly with respect to $t \in [-T, T]$ for any fixed $t > 0$ (under the conditions of the theorem, obviously, we can continue the specified functions $a$ and b to these functions).

Consider the difference scheme

$$k^{-1}(z_{j+1,n} - z_{j,n}) + D_+^2 D_- z_{j+1,n} + a_{j,n} z_{j+1,n} + b_{j,n} D_0 z_{j+1,n} = 0, \tag{11}$$

specified on a net consisting of points $(t_j, x_n)$, $t_j = jk$, $x_n = nh$, where $k > 0$ and $h > 0$ are steps of the net, while $j, n \in \mathbf{Z}$; here $z_{j,n} = z(t_j, x_n)$. In (11), $D_0$, $D_+$, and $D_-$ denote the following difference operators, which act with respect to the variable $x_n$:

$$D_0 \rho(x_n) = (2h)^{-1}[\rho(x_{n+1}) - \rho(x_{n-1})],$$
$$D_+ \rho(x_n) = h^{-1}[\rho(x_{n+1}) - \rho(x_n)], \quad D_- \rho(x_n) = h^{-1}[\rho(x_n) - \rho(x_{n-1})].$$

We introduce two norms on net functions $z = z(x_n)$:

$$|z|_h^2 = \sum_n h[z(x_n)]^2, \quad |||z|||_h^2 = |z|_h^2 + |xD_0 z|_h^2 + |D_+ z|_h^2.$$

Moreover, for functions $z \in C^\infty(\mathbf{R}_x)$ we set

$$|||z|||^2 = \|z\|^2 + \|xz_x\|^2 + \|z_{xxx}\|^2.$$

The following lemma, whose proof is analogous to the one given in [2], means that difference scheme (11) is solvable and stable with respect to the norm $|||\cdot|||_h$.

Lemma 1. For any K > 0 there exist positive constants $t_K$ and L such that if $|||z_0|||<K$ and k is sufficiently small, then scheme (11) is solvable with respect to $z^j = z(t_j, \cdot)$ for $0 \leq t_j \leq t_K$, and $|||z(t_j, \cdot)|||<L$ in this case.

Now we need to employ norms with weights $\langle x\rangle^N$ on net functions, where $\langle x\rangle = (1+|x|^2)^{1/2}$. The proof of the following lemma is analogous to the proof in [2].

**Lemma 2.** For any $N \geq 0$ and any integer $n \geq 0$ there exist $C > 0$, $h_0 > 0$, $k_0 > 0$ such that for $0 \leq t_j \leq t_K$, $h < h_0$, $k < k_0$ we have the bound

$$\|\langle x\rangle^N D_+^n z(t_j, \cdot)\|_h < C.$$

As in [2], local existence of a solution can be derived from Lemmas 1 and 2. By analogy with [1], we can prove the following a priori estimate.

Lemma 3. For any C > 0 and T > 0 there exists a K > 0 such that if $z = z(t, x)$ is a solution of problem (8) with initial condition $z_0$, for which $|||z_0|||<C$, then $|||z(t, \cdot)|||<K$ for $0 \leq t \leq T$.

From this and from the local existence theorem, as in [2], we obtain that there exists a solution of problem (8) that is defined for all $t \in [0, T]$.

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