THEOREMS ON THE COINCIDENCE OF THE SPECTRA OF PSEUDODIFFERENTIAL ALMOST-PERIODIC OPERATORS IN THE SPACES $L^2(\mathbf{R}^n)$ AND $B^2(\mathbf{R}^n)$

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Pseudodifferential almost-periodic (p.a.) operators were introduced in [1], in which a certain special Π_∞-representation of the algebra of these operators was studied, as well as related questions on the index and invertibility in the space $L^2(\mathbf{R}^n)$. Analogous questions were considered in [2] for differential operators, but in Hölder spaces of p. a. functions and in spaces of p. a. Stepanov functions. In [3] the Schrödinger operator was studied in the space $B^2(\mathbf{R}^n)$ of p. a. Besicovitch functions. This last space, which is Hilbert but nonseparable, is extremely useful for the study of p. a. operators. This was shown by the author in [4] (cf., also [5]), where the space $B^2(\mathbf{R}^n)$ was used to construct an Hilbertian scale of spaces of Sobolev type (and other analogous scales) and various theorems were proved on the action of p. a. operators in spaces of p. a. functions.

The aim of this paper is to prove a theorem on coincidence of the spectra of a pseudodifferential p. a. operator in the spaces $L^2(\mathbf{R}^n)$ and $B^2(\mathbf{R}^n)$.

We remark that the characters of the spectra of an operator in these spaces can be different: e.g., the Laplace operator, which has a continuous spectrum in $L^2(\mathbf{R}^n)$, has in the space $B^2(\mathbf{R}^n)$ an orthogonal basis of characteristic functions.

This paper consists of five sections and an Appendix. In § 1 we introduce the necessary definitions and notations. In §§ 2 and 3 we prove two important auxiliary theorems concerning the relation between quadratic forms of operators in the spaces $L^2(\mathbf{R}^n)$ and $B^2(\mathbf{R}^n)$. In § 4 we sketch a theory of formally adjoint pairs of operators in Hilbert space and its application to concrete operators. Finally, § 5 contains the formulation and proof of the principal theorem.

In the Appendix we present a variant of the Kronecker – Weyl theorem, used in § 3.

§ 1. Definitions and Preparatory Considerations

The symbol $\mathbf{R}^n$ will denote the n-dimensional coordinate vector space over the field R of real numbers. If $x \in \mathbf{R}^n$, then $x = (x_1, \ldots, x_n)$, and we define $|x|_\infty = \sup_{1\le j\le n} |x_j|$. The symbol **Z** will denote the ring of integers. As usual we define $D_j = i^{-1}\partial/\partial x_j$, $D = (D_1, \ldots, D_n)$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ $(\alpha_j \in \mathbf{Z},\ \alpha_j \ge 0)$. If α is a multi-index, then by definition $|\alpha| = \alpha_1 + \ldots + \alpha_n$. If $T > 0$, then the symbol K_T will denote the following cube in $\mathbf{R}^n$:

$$K_T = \{x : x \in \mathbf{R}^n,\ |x|_\infty \le T/2\}.$$

The volume of the cube K_T, using the Lebesgue measure in $\mathbf{R}^n$ (denoted dx) is equal to T^n.

A set $E \subset \mathbf{R}^n$ is said to be relatively dense if there exists a $L > 0$ such that $(z + K_L) \cap E \neq \emptyset$ for any $z \in \mathbf{R}^n$ (here $z + K_L$ denotes the cube K_L, translated by z).

The symbol $CAP(\mathbf{R}^n)$ will denote the Banach space of uniformly p. a. functions, i.e., functions such that for any $\varepsilon > 0$ there is a relatively dense set in $\mathbf{R}^n$ of ε-almost-periods [a vector $\tau \in \mathbf{R}^n$ is called an ε-almost-period for the function $f(x)$ if $\sup_{x \in \mathbf{R}^n} |f(x + \tau) - f(x)| < \varepsilon$]. An equivalent definition of a uniformly p. a. function: $f \in CAP(\mathbf{R}^n)$, if f is bounded and the set of translations $\{f(\cdot + \tau)\}_{\tau \in \mathbf{R}^n}$ is precompact in the topology of uniform convergence. If in this case $D^\gamma f \in CAP(\mathbf{R}^n)$ for any multi-index γ with $|\gamma| \le m$, then we shall say that $f \in CAP^m(\mathbf{R}^n)$.

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We remark that $CAP^m(\mathbf{R}^n)$ is a Banach algebra. We define $CAP^\infty(\mathbf{R}^n) = \bigcap_m CAP^m(\mathbf{R}^n)$.

By the theorem of the mean for $f \in CAP(\mathbf{R}^n)$, the limit

$$M\{f\} = \lim_{T\to\infty} \frac{1}{T^n} \int_{z+K_T} f(x)\, dx \tag{1.1}$$

exists uniformly with respect to $z \in \mathbf{R}^n$ and does not depend on z (the proof is analogous to the case n = 1, which is considered, e.g., in the monograph [6]). This limit is also uniform with respect to any compact family of functions from $CAP(\mathbf{R}^n)$. We note that the compactness of a set $\mathcal{F} \subset CAP(\mathbf{R}^n)$ is equivalent to the uniform boundedness, equicontinuity and equi-almost-periodicity of all functions $f \in \mathcal{F}$. The definitions of all of these concepts follow the one-dimensional case (cf., [6]). We shall only show that the equi-almost-periodicity of a set $\mathcal{F} \subset CAP(\mathbf{R}^n)$ means that for any $\varepsilon > 0$ there exists a relatively dense $E \subset \mathbf{R}^n$ whose elements are generalized ε-almost-periods uniformly for all $f \in \mathcal{F}$.

We now define in $CAP(\mathbf{R}^n)$ the scalar product

$$(f, g)_B = M\{f(x)\overline{g(x)}\}. \tag{1.2}$$

[The notation (f, g) will be needed for the usual scalar product $L^2(\mathbf{R}^n)$.] We introduce the Hilbert space $B^2(\mathbf{R}^n)$, which is the completion of $CAP(\mathbf{R}^n)$ in the Hilbert norm, induced by the scalar product (1.2). In the space $B^2(\mathbf{R}^n)$ the exponentials $e^{i\xi\cdot x}$ form an orthogonal basis, where $\xi \in \mathbf{R}^n$, $\xi \cdot x = \xi_1 x_1 + \ldots + \xi_n x_n$. Linear combinations of such exponentials, also known as trigonometric polynomials, form a linear space, denoted $\mathrm{Trig}(\mathbf{R}^n)$, and it is dense in $B^2(\mathbf{R}^n)$ and $CAP(\mathbf{R}^n)$.

Now let $a(x, \xi) \in C^\infty(\mathbf{R}^{2n})$, m, ρ, and δ be certain real numbers. We shall write $a(x, \xi) \in APS^m_{\rho,\delta}$ if for any multi-indices α and β the function $\partial_\xi^\alpha \partial_x^\beta a(\cdot, \xi)$ is a continuous function on $\mathbf{R}^n_\xi$ with values in $CAP(\mathbf{R}^n_x)$, satisfying the estimates

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \le C_{\alpha\beta}(1 + |\xi|)^{m - \rho|\alpha| + \delta|\beta|}. \tag{1.3}$$

Throughout the following we assume that $0 \le \delta < \rho \le 1$.

We define the operator $A = \mathrm{Op}(a)$ with symbol $a(x, \xi)$ by the formula

$$Au(x) = (2\pi)^{-n} \int e^{i\xi\cdot(x-y)} a(x, \xi) u(y)\, dy\, d\xi, \tag{1.4}$$

which can easily be made meaningful (on the right side of (1.4) we have an oscillating integral), if $u \in CAP^\infty(\mathbf{R}^n)$ (cf., e.g., [7-10] or [4]). The set of all operators $A = \mathrm{Op}(a)$ with $a \in APS^m_{\rho,\delta}$ will be denoted $APL^m_{\rho,\delta}$. The corresponding pseudodifferential operators will be called almost-periodic. An operator $A \in APL^m_{\rho,\delta}$ will map either of the spaces $CAP^\infty(\mathbf{R}^n)$ or $S(\mathbf{R}^n)$ into itself (this latter is the Schwarz space of functions all of whose derivatives decrease as $|x| \to \infty$ more rapidly than any power of $|x|$). It has a formally adjoint operator $A^+ \in APL^m_{\rho,\delta}$ (in $L^2(\mathbf{R}^n)$ and $B^2(\mathbf{R}^n)$ one and the same – cf., [4]) and therefore has a closure in these spaces. An operator $A \in APL^0_{\rho,\delta}$ can be continued to a continuous operator in the spaces $L^2(\mathbf{R}^n)$ (cf., [8]) and $B^2(\mathbf{R}^n)$ (cf., [4]).

We, furthermore, recall the definitions of the classes $APHS^{m,m_0}_{\rho,\delta}$ and $APHL^{m,m_0}_{\rho,\sigma}$ of hyperelliptic symbols and operators (cf., [4]). We say that $a(x, \xi) \in APHS^{m,m_0}_{\rho,\sigma}$ if $a(x, \xi) \in APS^m_{\rho,\delta}$ and if there exists an $R > 0$ such that for $|\xi| \ge R$ the following estimates hold

$$|a(x, \xi)| \ge C|\xi|^{m_0}, \tag{1.5}$$

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \le C_{\alpha\beta} |a(x, \xi)|\, |\xi|^{-\rho|\alpha| + \delta|\beta|}. \tag{1.6}$$

The symbol $APHL^{m,m_0}_{\rho,\delta}$ will denote the corresponding class of operators. If $A \in APHL^{m,m_0}_{\rho,\delta}$, $m_0 > 0$, and $A = A^+$, then A is essentially adjoint in $L^2(\mathbf{R}^n)$ and in $B^2(\mathbf{R}^n)$, if a priori it is considered on $C_0^\infty(\mathbf{R}^n)$ and $\mathrm{Trig}(\mathbf{R}^n)$, respectively (cf., [11, 4]).

§ 2. The First Theorem on the Connection between Quadratic Forms

We introduce a family of real-valued functions $\varphi_T(x) \in C_0^\infty(\mathbf{R}^n)$, $T \ge 1$, satisfying for some $\varkappa \in \mathbf{R}$ the following conditions:

$$\varphi_T(x) = 1 \quad \text{for} \quad |x|_\infty \leq T; \tag{2.1}$$

$$\varphi_T(x) = 0 \quad \text{for} \quad |x|_\infty \geq T + T^\varkappa; \tag{2.2}$$

$$|D^\gamma \varphi_T(x)| \leq C_\gamma T^{-\varkappa|\gamma|}, \tag{2.3}$$

where γ is a multi-index, and C_γ does not depend on T. The existence of the family $\varphi_T(x)$ is easily verified for any fixed $\varkappa \in R$. Throughout what follows $\varkappa$ will be fixed and will satisfy the condition

$$0 < \varkappa < 1. \tag{2.4}$$

THEOREM 2.1. Suppose that the number $\varkappa$ is fixed, and that a family of functions $\varphi_T(x)$ satisfying conditions (2.1)-(2.4) is given. Let $A_j \in APL^{m_j}_{\rho,\delta}$, $j = 1, 2$, $0 \leq \delta < \rho \leq 1$, and suppose that $u_j \in CAP^\infty(R^n)$, $j = 1, 2$. Then

$$(A_1 u_1, A_2 u_2)_B = \lim_{T\to\infty} \frac{1}{T^n} (A_1(\varphi_T u_1), A_2(\varphi_T u_2)). \tag{2.5}$$

Proof. We first note that (2.5) may be rewritten in the form

$$(A_2^+ A_1 u_1, u_2)_B = \lim_{T\to\infty} \frac{1}{T^n} (A_2^+ A_1(\varphi_T u_1), \varphi_T u_2).$$

Defining $A_2^+ A_1 = A$, we see from a theorem on the composition of operators and adjoint operators from [4] that $A \in APL^{m_1+m_2}_{\rho,\delta}$. Standard polarization arguments allow us to limit ourselves to the case $u_1 = u_2 = u$. Therefore, it is sufficient to prove that

$$(Au, u)_B = \lim_{T\to\infty} \frac{1}{T^n} (A(\varphi_T u), \varphi_T u), \tag{2.6}$$

where $A \in APL^m_{\rho,\delta}$, $u \in CAP^\infty(R^n)$.

We note first that the volume of the set $\{x : T \leq |x|_\infty \leq T + T^\varkappa\}$ does not exceed $CT^{n-1+\varkappa}$, and since $\varkappa < 1$ and $|\varphi_T(x)| \leq C_0$, where C_0 does not depend on T, it follows that for $f \in CAP(R^n)$

$$M\{f\} = \lim_{T\to\infty} \frac{1}{T^n} \int_{R^n} \varphi_T(x) f(x) dx. \tag{2.7}$$

Therefore,

$$(Au, u)_B = \lim_{T\to\infty} \frac{1}{T^n} (Au, \varphi_T u). \tag{2.8}$$

Equating (2.6) and (2.8) we see that it remains to be proved that

$$\lim_{T\to\infty} \frac{1}{T^n} (A(1 - \varphi_T) u, \varphi_T u) = 0$$

for

$$\lim_{T\to\infty} \frac{1}{T^n} (u, (1 - \varphi_T) A(\varphi_T u)) = 0.$$

Substituting A^+ for A, we see that it is sufficient to prove the relation

$$\lim_{T\to\infty} \frac{1}{T^n} ((1 - \varphi_T) A(\varphi_T u), u) = 0. \tag{2.9}$$

Denoting the expression under the limit sign by I(T), we obtain

$$I(T) = \frac{1}{T^n} ((1 - \varphi_T) A(\varphi_T u), u) = \frac{1}{T^n} \int (1 - \varphi_T(x)) a(x, \xi) e^{i\xi\cdot(x-y)}$$
$$\varphi_T(y) u(y) \overline{u(x)}\, dy d\xi dx = \frac{1}{T^n} \int (1 - \varphi_T(x)) e^{i\xi\cdot(x-y)} (1 + |x - y|^2)^{-N}$$
$$(1 - \Delta_\xi)^N \left[a(x, \xi)(1 + |\xi|^2)^{-M}\right] (1 - \Delta_y)^M [\varphi_T(y) u(y)] \overline{u(x)}\, dy dx d\xi.$$

If A were a differential operator, then the continuation of the proof would be obvious from (2.7), since the volume of the region, where $(1 - \varphi_T)\varphi_T \neq 0$, does not exceed $CT^{n-1+\varkappa}$, while the function $(1 - \varphi_T) \times A(\varphi_T u)$ is bounded. Therefore, the idea of the calculations below consists of using estimates that give the pseudolocal character of the operator A.

To begin let $\psi(x) \in C_0^\infty(R^n)$, $\psi(x) = 1$ for $|x| \leq 1$, $\psi(x) = 0$ for $|x| \geq 2$. We choose $\delta > 0$ and divide the interval I(T) into a sum of two intervals $I(T) = I_1(T) + I_2(T)$, where

$$I_1(T) = \frac{1}{T^n} \int \psi((x - y) \cdot T^{-\delta}) (1 - \varphi_T(x)) e^{i\xi\cdot(x-y)} (1 + |x - y|^2)^{-N}$$
$$\times (1 - \Delta_\xi)^N \left[a(x, \xi)(1 + |\xi|^2)^{-M}\right] (1 - \Delta_y)^M [\varphi_T(y) u(y)] \overline{u(x)}\, dy d\xi dx,$$

and $I_2(T)$ differs from $I_1(T)$ in the fact that in place of $\psi((x - y) \cdot T^{-\delta})$ under the interval we have $[1 - \psi((x - y) \cdot T^{-\delta})]$.

We choose δ such that the following conditions hold:

$$0 < \delta < \varkappa, \tag{2.10}$$

$$\varkappa + \delta n < 1. \tag{2.11}$$

We first consider the integral $I_1(T)$. The expression under the integral sign has for $m - 2M \leq -n - 1$ the bound $C(1 + |\xi|)^{-n-1}$, so that the integral with respect to ξ is bounded. In addition $|x - y| \leq 2T^\delta$, $|x|_\infty \geq T$, and $|y|_\infty \leq T + T^\varkappa$, so by (2.10) we have $T \leq |x|_\infty \leq T + 3T^\varkappa$ and the volume of the region of integration with respect to x and y does not exceed $CT^{n-1+\varkappa+n\delta}$. Since $n - 1 + \varkappa + n\delta < n$ by (2.11), it follows that $I_1(T) \to 0$ for $T \to +\infty$.

We now consider the integral $I_2(T)$. In this integral $|x - y| \geq T^\delta$. Integrating by parts we obtain

$$(1 + |x|^2)^{-L} (1 - \Delta_\xi)^L e^{i\xi\cdot x} = e^{i\xi\cdot x},$$

which guarantees the absolute convergence of the integral with respect to x. We obtain

$$I_2(T) = \sum_{|\alpha+\beta| \leq 2L} C_{\alpha\beta} I_{\alpha\beta}(T),$$

where α, β are multi-indices and

$$I_{\alpha\beta}(T) = \frac{1}{T^n} \int \left[1 - \psi((x - y) \cdot T^{-\delta})\right] (1 - \varphi_T(x))(1 + |x|^2)^{-L}$$
$$\times (1 + |x - y|^2)^{-N} e^{i\xi\cdot(x-y)} D_\xi^\alpha (1 - \Delta_\xi)^N \left[a(x, \xi)(1 + |\xi|^2)^{-M}\right] \; y^\beta (1 - \Delta_y)^M [\varphi_T(y) u(y)] \overline{u(x)}\, dy d\xi dx.$$

We choose fixed numbers M and L such that for $m - 2M \leq -n - 1$, $2L \geq n + 1$. We note that in the expression under the integral sign (2.12) $|y|_\infty \leq T + T^\varkappa \leq 2T$, therefore it is bounded by

$$C(1 + |x - y|^2)^{-N} (1 + |x|)^{-n-1} (1 + |\xi|)^{-n-1} T^{2L},$$

which in turn may be estimated by

$$C(1 + |x|)^{-n-1} (1 + |\xi|)^{-n-1} T^{2L-2N\delta}.$$

Taking into account the fact that both regions of integration with respect to y have volume no greater than $C(2T)^n$, we obtain

$$|I_{\alpha\beta}(T)| \leq CT^{2L-2N\delta+n}.$$

Choosing N so that $2N\delta > 2L + n$, we obtain

$$I_{\alpha\beta}(T) \to 0 \quad \text{for} \quad T \to +\infty,$$

from which it follows that $I_2(T) \to 0$ for $T \to +\infty$.

§ 3. A Second Theorem on the Connection between Quadratic Forms

Throughout this section the symbol Γ will denote a certain directed set, a partially ordered set with an order relation $\geq$ satisfying

$$\forall \gamma', \gamma'' \in \Gamma \;\exists \gamma = \gamma(\gamma', \gamma'') \in \Gamma \; (\gamma \geq \gamma', \gamma \geq \gamma'').$$

If M is a set, then the mapping $f : \Gamma \to M$ will be called a net of points in M. The image of a point γ under this mapping will be denoted f_γ and the entire net will be written in the form $\{f_\gamma\}_{\gamma \in \Gamma}$. If M is a topological space, and $\{f_\gamma\}_{\gamma \in \Gamma}$ is a net of its points with $f \in M$, then the relation $\lim_\Gamma f_\gamma = f$ means that for any neighborhood U of the point f there exists $\gamma_0 = \gamma_0(U)$ such that $f_\gamma \in U$ for $\gamma \geq \gamma_0$, $\gamma \in \Gamma$.

Let $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_N \subset \ldots$ be any increasing sequence of precompact subsets $\mathcal{F}_j \subset CAP(R^n)$. We shall use a net $\{\psi_\gamma\}_{\gamma \in \Gamma}$ of functions $\psi_\gamma \in CAP(R^n)$ such that the following conditions hold:

1) $\psi_\gamma(x) \geq 0$ for all $x \in R^n$, where $\psi_\gamma(0) > 0$;

2) $\forall \varepsilon > 0\ \forall N\ \exists \gamma_0 = \gamma_0(\varepsilon, N)\ \forall \gamma \geq \gamma_0\ \forall \tau \in R^n$;

it follows from the condition $\psi_\gamma(\tau) \neq 0$ that τ is a generalized ε-almost-period for all $f \in \mathcal{F}_N$;

3) if we denote $g_\gamma(x) = M_y\{\psi_\gamma(y - x)\psi_\gamma(y)\}$, then for any function $w \in S(R^n)$ the following relation holds:

$$\lim_\Gamma \int_{R^n} w(x) g_\gamma(x)\,dx = w(0) \tag{3.1}$$

[the functions $g_\gamma(x) \in CAP(R^n)$ form a generalized δ-shaped sequence for $S(R^n)$];

4) $\exists L > 0\ \exists C > 0\ \forall \gamma \in \Gamma$

$$\int (1+|x|)^{-L} g_\gamma(x)\,dx \leqslant C. \tag{3.2}$$

<u>LEMMA 3.1.</u> For any increasing sequence of precompact subsets $\mathcal{F}_j \subset CAP(R^n)$, there exists a net $\{\psi_\gamma\}_{\gamma \in \Gamma}$ of functions $\psi_\gamma \in CAP(R^n)$ satisfying conditions 1)–4).

We shall leave the proof of this lemma until the end of the section.

Now suppose that operators $A_j \in APL^{m_j}_{\rho,\delta}$, $j = 1, 2$, $0 \leq \delta < \rho \leq 1$, are given. We define $A = A_2^+ A_1$ and let $A = Op(a)$, where $a \in APS^m_{\rho,\delta}$, $m = m_1 + m_2$. We choose a fixed number $m' > m$ and define

$$\mathcal{F}_N = \{[\partial_\xi^\alpha \partial_x^\beta a(\cdot,\xi)](1+|\xi|)^{-m'+\rho|\alpha|-\delta|\beta|}\}_{\substack{\xi \in R^n,\\ |\alpha+\beta| \leqslant N}} \tag{3.3}$$

(ξ, α, and β are parameters). It is easy to verify that the sets $\mathcal{F}_N$ are precompact in $CAP(R^n)$. On them we construct a net of functions $\psi_\gamma(x) \in CAP(R^n)$ satisfying conditions 1)–4).

<u>THEOREM 3.1.</u> Let $v_j \in S(R^n)$, $A_j \in APL^{m_j}_{\rho,\delta}$, $j = 1, 2$; suppose that sets $\mathcal{F}_N \subset CAP(R^n)$ have been selected according to formula (3.3), and that a net $\{\psi_\gamma\}_{\gamma \in \Gamma}$ is given that satisfies conditions 1)–4). We define

$$v_{j,\gamma}(x) = \int v_j(y)\psi_\gamma(x-y)\,dy.$$

Then

$$v_{j,\gamma}(x) \in CAP^\infty(R^n) \text{ and } \lim_\Gamma (A_1 v_{1,\gamma}, A_2 v_{2,\gamma})_B = (A_1 v_1, A_2 v_2).$$

<u>Proof.</u> It is clear that it is sufficient to prove the relation

$$\lim_\Gamma (Av_\gamma, v_\gamma)_B = (Av, v), \tag{3.4}$$

where $A = Op(a) \in APL^m_{\rho,\delta}$, $v \in S(R^n)$, the functions $\psi_\gamma(x)$ are constructed on a sequence of sets $\mathcal{F}_N$, defined by (3.3), and

$$v_\gamma(x) = \int v(y)\psi_\gamma(x-y)\,dy. \tag{3.5}$$

It follows in the obvious way from (3.5) that $v_\gamma(x) \in CAP(R^n)$. Furthermore, rewriting (3.5) in the form

$$v_\gamma(x) = \int v(x-y)\psi_\gamma(y)\,dy,$$

we see that for any multi-index α

$$D^\alpha v_\gamma(x) = \int D^\alpha v(x-y)\psi_\gamma(y)\,dy = \int (D^\alpha v)(y)\psi_\gamma(x-y)\,dy,$$

from which it follows that $v_\gamma(x) \in CAP(R^n)$.

We shall now show that

$$\lim_\Gamma M\{|v_\gamma(x)|^2\} = (v, v) \tag{3.6}$$

(this is a particular case of (3.4), when A is the unit operator). We have

$$M\{v_\gamma(x)\overline{v_\gamma(x)}\} = \int v(y)\overline{v(z)} M_x\{\psi_\gamma(x-y)\psi_\gamma(x-z)\}\,dydz. \tag{3.7}$$

Noting that $M_x\{\psi_\gamma(x - y)\psi_\gamma(x - z)\} = M_x\{\psi_\gamma(x - (y - z))\psi_\gamma(x)\} = g_\gamma(y - z)$ and making the change of variable $y - z = t$ in the integral (3.7), we obtain

$$M\{|v_\gamma|^2\} = \int v(z+t)\overline{v(z)} g_\gamma(t)\,dzdt = \int w(t) g_\gamma(t)\,dt,$$

where $w(t) = \int v(z + t)\overline{v(z)}dz \in S(R^n)$, with $w(0) = (v, v)$. It is now clear that (3.6) follows from (3.1). We note further that a polarization applied to (3.6) gives for any $u, v \in S(R^n)$

$$\lim_\Gamma (u_\gamma, v_\gamma)_B = (u, v).$$

Now setting $u = Av$, we obtain

$$\lim_\Gamma ((Av)_\gamma, v_\gamma)_B = (Av, v). \tag{3.8}$$

We introduce the difference

$$d_\gamma(x) = Av_\gamma(x) - (Av)_\gamma(x).$$

Equating (3.8) and (3.4), we see that to verify (3.4) it is sufficient to prove that

$$\lim_\Gamma (d_\gamma, v_\gamma)_B = 0. \tag{3.9}$$

By definition we have

$$Av_\gamma(x) = (2\pi)^{-n}\int e^{i\xi\cdot(x-z)} a(x,\xi) v(z-y)\psi_\gamma(y)\,dydzd\xi;$$

$$(Av)_\gamma(x) = (2\pi)^{-n}\int e^{i\xi(x-y-z)} a(x-y,\xi) v(z)\psi_\gamma(y)\,dzd\xi dy = (2\pi)^{-n}\int e^{i\xi\cdot(x-z)} a(x-y,\xi) v(z-y)\psi_\gamma(y)\,dzd\xi dy$$

(the integrals must be understood as oscillating).

From this

$$d_\gamma(x) = (2\pi)^{-n}\int e^{i\xi(x-z)}[a(x-y,\xi) - a(x,\xi)] v(z-y)\psi_\gamma(y)\,dzd\xi dy. \tag{3.10}$$

Transforming the oscillatory integral (3.10) into a convergent integral by means of a standard integration by parts, we obtain

$$\begin{aligned} d_\gamma(x) = (2\pi)^{-n}\int e^{i\xi\cdot(x-z)}(1+|x-z|^2)^{-M_1}(1-\Delta_\xi)^{M_1}[a(x-y,\xi) \\ - a(x,\xi))(1+|\xi|^2)^{-M_2}](1-\Delta_z)^{M_2} v(z-y)\psi_\gamma(y)\,dzd\xi dy, \end{aligned} \tag{3.11}$$

where the natural numbers M_1 and M_2 are chosen so that

$$2M_1 > n,\ 2M_2 > m+n. \tag{3.12}$$

We note that condition 2), combined with (3.3), implies that for any $\varepsilon > 0$ there exists $\gamma_0 \in \Gamma$ such that $\gamma \geq \gamma_0$

$$|(1-\Delta_\xi)^{M_1}[(a(x-y,\xi) - a(x,\xi))(1+|\xi|^2)^{-M_2}]|\psi_\gamma(y) \leqslant \varepsilon(1+|\xi|)^{m'-2M_2}\psi_\gamma(y).$$

Recalling, in addition, $v \in S(R^n)$, we obtain for $2M_3 > n$ and for M_1 and M_2, satisfying (3.12),

$$|d_\gamma(x)| \leqslant \varepsilon C_{M_1,M_2,M_3}\int (1+|x-z|)^{-2M_1}(1+|z-y|)^{-2M_3}(1+|\xi|)^{m'-2M_2}\psi_\gamma(y)\,dydzd\xi,$$

from which, choosing $M_1 = M_3$ and taking into account the obvious inequality

$$1+|x-y| \leqslant (1+|x-z|)(1+|z-y|),$$

we obtain

$$|d_\gamma(x)| \leqslant \varepsilon C_M \int (1+|x-y|)^{-M}\psi_\gamma(y)\,dy$$

or

$$|d_\gamma(x)| \leqslant \varepsilon C_M \int (1+|y|)^{-M}\psi_\gamma(x-y)\,dy, \tag{3.13}$$

where M is any even number satisfying $M > n$.

It follows from (3.13) that

$$|(d_\gamma, v_\gamma)_B| = |M_x\{d_\gamma(x)\overline{v_\gamma(x)}\}| \leqslant M_x\{|d_\gamma(x)||v_\gamma(x)|\} \leqslant \varepsilon C_{M,P}\int (1+|y|)^{-M}(1+|z|)^{-P} M_x\{\psi_\gamma(x-y)\psi_\gamma(x-z)\}\,dydz,$$