

DE RHAM THEOREM FOR EXTENDED L^2 -COHOMOLOGY

M.A.SHUBIN¹

Northeastern University

Dedicated to Selim Grigor'evich Krein on the occasion of his 80th birthday

ABSTRACT. We prove an analogue of the de Rham theorem for the extended L^2 -cohomology introduced by M.Farber [Fa]. This is done by establishing that the de Rham complex over a compact closed manifold with coefficients in a flat Hilbert bundle E of \mathcal{A} -modules over a finite von Neumann algebra \mathcal{A} is chain-homotopy equivalent in the sense of [GS] (i.e. with bounded morphisms and homotopy operators) to a combinatorial complex with the same coefficients. This is established by using the Witten deformation of the de Rham complex. We also prove that the de Rham complex is chain-homotopy equivalent to the spectrally truncated de Rham complex which is also finitely generated.

INTRODUCTION

L^2 -cohomology is a natural tool for constructing invariants of non-compact manifolds. To define it we need an extra structure of the manifold near infinity, e.g. a Riemannian metric or a triangulation, or rather a quasiisometry class of metrics or a class of triangulations which are compatible with a uniform structure. Sometimes such an extra structure naturally exists, e.g. if the manifold is the open set of all regular points of an algebraic variety, or if it has a proper cocompact action of a discrete group (e.g. if it is a regular covering of a compact manifold).

Assume that we have such a manifold Y and that an admissible triangulation T of it is given. Then we can define Hilbert spaces of L^2 -cochains $C_{(2)}^i = C_{(2)}^i(T, Y)$ and they form a complex

$$\dots \longrightarrow C_{(2)}^{i-1} \xrightarrow{d_{i-1}} C_{(2)}^i \xrightarrow{d_i} C_{(2)}^{i+1} \longrightarrow \dots, \quad (0.1)$$

where d_i are bounded linear operators.

There are two possible ways to define the L^2 -cohomology of this complex. One of them is just to form the usual cohomology

$$L^2 H^i(T, Y) = \text{Ker } d_i / \text{Im } d_{i-1},$$

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ignoring topology. But then we have to face the fact that these cohomology spaces are infinite-dimensional and have no natural Hausdorff topology.

Another way is to form reduced cohomology

$$L^2H^i(T, Y) = \text{Ker } d_i / \overline{\text{Im } d_{i-1}},$$

where the bar over $\text{Im } d_{i-1}$ means its closure in the Hilbert space $C^i_{(2)}$. However, this leads to a big loss of information.

It was first noticed in [NS] that for covering manifolds topological information can be extracted by considering the behavior of the spectra of the Laplacians of the complex (0.1) near 0. In particular some numbers characterizing the behavior of the spectrum near zero were defined in [NS1] (they were later called Novikov-Shubin invariants). It was proved in [GS] that they are in fact homotopy invariants. More details about these invariants can be found in [E, E1, GS, LL]. In particular it follows from [GS] that these invariants are well-defined for arbitrary regular coverings of finite CW-complexes.

M.Farber [Fa] discovered a way to approach the spectrum-near-zero phenomenon through a new cohomology theory. It is convenient to describe his approach in a more general context. Namely, let \tilde{X} be a manifold with a free action of a discrete group Γ so that the quotient manifold $X = \tilde{X}/\Gamma$ is compact. Then L^2 -functions or L^2 -forms on \tilde{X} can be considered as sections of an infinite-dimensional vector bundle over X . The fiber of this bundle is a finitely-generated Hilbert module over the finite von Neumann algebra \mathcal{A} associated with Γ .

M.Farber actually considered even more general situation of an arbitrary Hilbert bundle over a finite CW-complex. The fiber of this bundle should be a finitely generated Hilbert \mathcal{A} -module, where \mathcal{A} is an arbitrary finite von Neumann algebra. In this setting M.Farber introduced an extended cohomology theory. It takes values in an abelian category $\mathcal{E}(\mathcal{A})$ which is obtained by applying a P.Freyd construction to the additive category $\mathcal{H}(\mathcal{A})$ of all finitely generated Hilbert \mathcal{A} -modules. The category $\mathcal{E}(\mathcal{A})$ has $\mathcal{H}(\mathcal{A})$ as the full subcategory of all projective objects.

Now a cochain complex in the category $\mathcal{H}(\mathcal{A})$ can be considered as a complex in $\mathcal{E}(\mathcal{A})$ and the corresponding cohomology is called *extended cohomology*. The extended cohomology objects belong to the category $\mathcal{E}(\mathcal{A})$.

The reduced L^2 -cohomology spaces appear then as the projective parts of the extended cohomology objects. But the extended cohomology objects contain also the torsion parts, which determine the Novikov-Shubin invariants. There are also some other numerical invariants determined by the torsion part of extended cohomology. One of them is the number of generators, which was used in [Fa] to improve the von Neumann version of the Morse inequalities from [NS].

The construction suggested in [Fa] used cochain complexes arising from a cell decomposition of a manifold or a more general polyhedron. Our purpose in this paper is to define the de Rham version of the extended L^2 -cohomology of a manifold and to understand the corresponding de Rham theorem. This would simplify calculations of extended L^2 -cohomology and make possible its applications in geometry and analysis.

In order to define the de Rham version of extended cohomology we will consider spaces of L^2 differential forms with values in a flat vector bundle E such that its fiber is a finitely generated Hilbert module over a finite von Neumann algebra \mathcal{A} . These forms constitute a twisted de Rham complex which is a complex of

Hilbert \mathcal{A} -modules, and the differentials are (generally unbounded) closed, densely defined linear operators. Unfortunately, the Hilbert \mathcal{A} -modules of forms are not finitely generated, and it does not seem possible to generalize the construction of the extended category [Fa] to include such complexes. In order to overcome this difficulty, we consider the homotopy type of the twisted de Rham complex with respect to chain-homotopy equivalence as studied in [GS, GS1] (i.e. with bounded morphisms and homotopy operators). Then we define *extended de Rham cohomology* as the extended cohomology of any finitely generated Hilbert \mathcal{A} -complex which is homotopy-equivalent to the de Rham complex. Such a finitely generated \mathcal{A} -complex is called a *finite approximation* of the de Rham complex. The result does not depend on the choice of approximation.

Now we have to connect the extended de Rham cohomology with extended combinatorial chain cohomology. We prove that they coincide and this is what we call the *de Rham theorem in extended cohomology*. In fact the de Rham complex is homotopy-equivalent to the corresponding combinatorial complex, so the combinatorial cochain complex is a finite approximation of the de Rham complex. This is proved with the use of the Witten deformation of the de Rham complex. It is important here that in the Witten approximation we have a gap separating the “small eigenvalues” from the rest of the spectrum (see [BFKM] and [Sh2]).

Another finite approximation of the de Rham complex can be obtained by truncating all the L^2 -form spaces by the spectral projections of the Laplacians. So the desired approximation is just the subcomplex corresponding to the “small eigenvalues”. It is more difficult here to establish that this “small eigenvalues” complex is really finitely generated. To do this we use uniform kernel estimates and factor decomposition of the fiber.

The de Rham type theorems which are established in this paper generalize two well-known theorems of this kind: a result of J.Dodziuk [D], who studied the reduced L^2 -cohomology, and a result of A.Efremov [E, E1] who proved the equality of the combinatorial and analytic Novikov-Shubin invariants.

W.Lück [Lü] suggested a more algebraic cohomological approach to the description of the spectra near zero which is equivalent to Farber’s approach. Lück’s technique can be used in the De Rham theory as well, though Farber’s approach seems more convenient.

After becoming acquainted with my idea of de Rham theory in extended cohomology M.Farber suggested a different approach to de Rham theory. His approach leads to a natural relation of extended de Rham theory with Čech cohomology and makes use of sheaf theory ([Fa1]).

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1. PRELIMINARIES

In this section we will briefly describe Farber’s extended cohomology theory [Fa].

Let \mathcal{A} be a finite von Neumann algebra. This means that \mathcal{A} is a von Neumann algebra which has a finite trace $\tau : \mathcal{A} \rightarrow \mathbb{C}$ which is faithful and normal. We will fix this trace and we will always assume it normalized, i.e. $\tau(1) = 1$. If \mathcal{A} happens to be a factor, then it has to be of type I_n or II_1 . More generally a finite von Neumann algebra \mathcal{A} is a direct integral of type I_n and II_1 factors. A definition of finite von Neumann algebra in terms of projections can be found in [T] (Definition 1.16, Ch.V, p.296).

Recall that \mathcal{A} can be represented as a weakly closed selfadjoint subalgebra of the algebra of all bounded linear operators in a Hilbert space. The operator norm induces a norm in \mathcal{A} which does not depend on the choice of the representation. We will refer to this norm as the C^* -norm in \mathcal{A} . The corresponding topology in \mathcal{A} will be called the *norm topology*.

The operation $B \rightarrow B^*$ (taking the adjoint operator) is a well-defined involution in \mathcal{A} .

Let $\ell^2(\mathcal{A})$ denote the Hilbert space obtained as the completion of \mathcal{A} with respect to the inner product given by the trace: $(a, b) = \tau(b^*a)$ for all $a, b \in \mathcal{A}$.

A *Hilbert module* over \mathcal{A} is a Hilbert space M together with a continuous left \mathcal{A} -module structure (here \mathcal{A} is considered with its norm topology) such that there exists an isometric \mathcal{A} -linear embedding of M into $\ell^2(\mathcal{A}) \otimes H$, for some Hilbert space H . A Hilbert module M is *finitely generated* if it admits an embedding $M \rightarrow \ell^2(\mathcal{A}) \otimes H$ as above with finite-dimensional H .

It is often more convenient to forget about the inner product in a Hilbert module M and consider this module up to an \mathcal{A} -linear topological isomorphism. In [Fa] the corresponding class is called *Hilbertian module*. In other words a Hilbertian module is a topological vector space M with a structure of a left (algebraic) \mathcal{A} -module such that the action of \mathcal{A} is continuous and there exists an inner product $(\ , \)$ on M which generates the topology of M and such that M together with $(\ , \)$ and with the \mathcal{A} -action is a Hilbert module. The difference between Hilbert and Hilbertian modules is important in categorical constructions but it is not important for us. Therefore we shall only deal with Hilbert \mathcal{A} -modules, though isomorphisms of Hilbert \mathcal{A} -modules will be understood as arbitrary \mathcal{A} -linear topological isomorphisms (not necessarily unitary ones).

Let $\mathcal{H}(\mathcal{A})$ denote the additive category whose objects are finitely generated left Hilbert \mathcal{A} -modules and whose morphisms are continuous \mathcal{A} -module homomorphisms. This category depends on the choice of the trace τ .

The P.Freyd construction [Fr] provides an extended category $\mathcal{E}(\mathcal{A})$ which is an abelian category, containing $\mathcal{H}(\mathcal{A})$ as a full subcategory. An *object* of the category $\mathcal{E}(\mathcal{A})$ is defined as a morphism $(\alpha : A' \rightarrow A)$ in the category $\mathcal{H}(\mathcal{A})$. Given a pair of objects $\mathcal{X} = (\alpha : A' \rightarrow A)$ and $\mathcal{Y} = (\beta : B' \rightarrow B)$ of $\mathcal{E}(\mathcal{A})$, a *morphism* $\mathcal{X} \rightarrow \mathcal{Y}$ in the category $\mathcal{E}(\mathcal{A})$ is an equivalence class of morphisms $f : A \rightarrow B$ of the category $\mathcal{H}(\mathcal{A})$ such that $f \circ \alpha = \beta \circ g$ for some morphism $g : A' \rightarrow B'$ in $\mathcal{H}(\mathcal{A})$. Two morphisms $f : A \rightarrow B$ and $f' : A \rightarrow B$ of $\mathcal{H}(\mathcal{A})$ represent the same morphism $\mathcal{X} \rightarrow \mathcal{Y}$ in $\mathcal{E}(\mathcal{A})$ iff $f - f' = \beta \circ F$ for some morphism $F : A \rightarrow B'$ of the category $\mathcal{H}(\mathcal{A})$. This defines an equivalence relation. The *composition* of morphisms is defined as the composition of the corresponding morphisms f in the category $\mathcal{H}(\mathcal{A})$.

Since $\mathcal{E}(\mathcal{A})$ is an abelian category, any cochain complex D in $\mathcal{H}(\mathcal{A})$ (or in $\mathcal{E}(\mathcal{A})$) has cohomology $\mathcal{H}^*(D)$ which is well-defined as a graded object of $\mathcal{E}(\mathcal{A})$. It is called the *extended cohomology* and it has all the standard cohomological properties.

In particular it is functorial, i.e. any chain map between cochain complexes $g : D_1 \rightarrow D_2$ in $\mathcal{H}(\mathcal{A})$ induces a morphism of their extended cohomologies $g^* : \mathcal{H}^*(D_1) \rightarrow \mathcal{H}^*(D_2)$.

Extended cohomology is homotopy invariant, i.e. the induced morphism g^* depends only on the homotopy class of g .

There is a long exact sequence

$$\dots \rightarrow \mathcal{H}^i(D_1) \rightarrow \mathcal{H}^i(D_2) \rightarrow \mathcal{H}^i(D_3) \rightarrow \mathcal{H}^{i+1}(D_1) \rightarrow \dots$$

corresponding to any short exact sequence

$$0 \rightarrow D_1 \rightarrow D_2 \rightarrow D_3 \rightarrow 0$$

of cochain complexes in $\mathcal{H}(\mathcal{A})$.

Now let us recall the definition of extended cohomology in a topological context.

Let Γ be a countable discrete group and M be a finitely generated Hilbert \mathcal{A} -module. Suppose that we have a right action of Γ on M by \mathcal{A} -automorphisms i.e. a representation $\rho : \Gamma^{op} \rightarrow GL_{\mathcal{A}}(M)$. Here $GL_{\mathcal{A}}(M)$ is the group of all (bounded) \mathcal{A} -automorphisms of M (not necessarily unitary), and Γ^{op} is the group which is obtained from Γ by introducing a new operation so that the product of α and β in Γ^{op} equals to the product $\beta\alpha$ in Γ . Following [BFKM] we will say in this case that M is a finitely generated Hilbert $(\mathcal{A}, \Gamma^{op})$ -module.

Note that by definition the actions of \mathcal{A} and Γ^{op} on M commute.

The simplest important example of this situation appears when $M = \ell^2(\Gamma)$, and $\mathcal{A} = \mathcal{N}(\Gamma)$ is the von Neumann algebra generated by the left translations by the elements $\gamma \in \Gamma$ in $\ell^2(\Gamma)$. There is also a natural right action of Γ^{op} in $\ell^2(\Gamma)$ by right translations. This action obviously commutes with the action of $\mathcal{N}(\Gamma)$. Therefore $\ell^2(\Gamma)$ has the structure of a finitely generated Hilbert $(\mathcal{N}(\Gamma), \Gamma^{op})$ -module.

Let X be a finite, connected CW-complex and Γ its fundamental group. Let \tilde{X} be the universal covering of X . The group Γ acts on \tilde{X} by deck transformations. Therefore for any finitely generated Hilbert $(\mathcal{A}, \Gamma^{op})$ -module M we can form the space $E = M \times_{\Gamma} \tilde{X}$. There is a natural projection $p : E \rightarrow X$ so that E becomes a flat Hilbert \mathcal{A} -bundle with standard fiber M and representation ρ giving the monodromy.

A flat Hilbert \mathcal{A} -bundle E with fiber M over a topological space X can also be defined by a 1-cocycle on X with values in $GL_{\mathcal{A}}(M)$. Namely, for a sufficiently fine open covering \mathcal{U} of X and any $(U, V) \in \mathcal{U} \times \mathcal{U}$ we should be given an element $g_{UV} \in GL_{\mathcal{A}}(M)$ so that the following conditions are satisfied:

- (a) $g_{UU} = 1$;
- (b) $g_{UV} \cdot g_{VW} = g_{UW}$ if $U \cap V \cap W \neq \emptyset$.

The total space of the bundle E is constructed then by identifying the points $(x, m) \in U \times M$ with $(x, g_{VU} \cdot m) \in V \times M$ in the disjoint union of the spaces $U \times M$ for $x \in U \cup V$.

Let $X' \subset X$. Then any flat Hilbertian bundle E over X has a naturally defined restriction $E' = E|_{X'}$ which is a flat Hilbertian bundle over X' .

Assume for a moment that X is a compact closed C^∞ -manifold. Denote by $C^\infty(X, E)$ and $L^2(X, E)$ the spaces of C^∞ -sections and L^2 -sections of E , respectively. It is easy to see that if we take $M = \ell^2(\Gamma)$ with the action of Γ^{op} by right translations as described above, then the space $L^2(X, \ell^2(\Gamma))$ is naturally identified with the space $L^2(\tilde{X})$ defined by a Γ -invariant measure (i.e. a measure lifted from X) with a smooth positive density. Under this identification the space $C^\infty(X, \ell^2(\Gamma))$ becomes the space $H^\infty(\tilde{X})$ which can be described as the intersection of all uniform Sobolev spaces on \tilde{X} (see [Sh1] for more details about these spaces) or as the space

of all $u \in C^\infty(\tilde{X})$ such that $Du \in L^2(\tilde{X})$ for every Γ -invariant differential operator D with C^∞ coefficients.

Let us return to the general case when X is a finite, connected CW-complex. Then the universal covering \tilde{X} has the natural structure of a CW-complex generated by the lifts of the cells of X . A flat bundle E as above defines an associated locally constant sheaf \underline{E} . The space of sections of \underline{E} over a given cell is isomorphic to the standard fiber M . Using the restriction maps in the usual way, we obtain a finitely generated Hilbert cochain \mathcal{A} -complex which we will denote by $(\mathcal{C}^\bullet, \delta)$. It can be also defined purely algebraically (see [Fa] for the definitions in this particular context).

The extended cohomology of $(\mathcal{C}^\bullet, \delta)$ does not depend on the choice of CW representation of X ([Fa]). We will call it the *extended cohomology of X* (with coefficients in E) and denote it by $\mathcal{H}^*(X, E)$.

In the case X is a compact closed manifold, we will actually use a special CW-complex which is associated with a generic Morse function as described in F.Laudenbach's appendix to [BZ] and also in [BFKM].

2. FREDHOLM COMPLEXES AND THEIR EXTENDED L^2 -COHOMOLOGY

We would like to generalize the definition of extended cohomology to a category of Fredholm complexes of Hilbert \mathcal{A} -modules with unbounded differentials (like the one considered in [GS]). The extended cohomology objects will still belong to the extended category $\mathcal{E}(\mathcal{A})$.

Let

$$C : \dots \rightarrow C^{i-1} \xrightarrow{d_{i-1}} C^i \xrightarrow{d_i} C^{i+1} \rightarrow \dots$$

be a sequence of finite length formed by Hilbert \mathcal{A} -modules and closed, densely defined linear operators d_i . It is called a Hilbert cochain \mathcal{A} -complex if the following two conditions are satisfied:

- (i) $d_{i+1}d_i = 0$ on $\text{Dom}(d_i)$ for all i (here $\text{Dom}(d_i)$ denotes the domain of d_i);
- (ii) For every i the operator d_i commutes with multiplication by any $a \in \mathcal{A}$ in the sense that $d_i a = a d_i$ on $\text{Dom}(d_i)$.

In particular we require that $a(\text{Dom}(d_i)) \subset \text{Dom}(d_i)$ for all $a \in \mathcal{A}$ and all i .

Assume that we have two Hilbert cochain \mathcal{A} -complexes C and C_1 with the differentials d_i, d'_i respectively. A *morphism of Hilbert cochain \mathcal{A} -complexes* $f : C \rightarrow C_1$ is a set of *bounded* linear \mathcal{A} -morphisms $f_i : C^i \rightarrow C_1^i$ such that $d'_i f_i = f_{i+1} d_i$ on $\text{Dom}(d_i)$. With this set of morphisms the Hilbert cochain \mathcal{A} -complexes form a category.

Two morphisms $f, g : C \rightarrow C_1$ of Hilbert \mathcal{A} -complexes are called *homotopic* if there exist bounded homotopy operators (\mathcal{A} -morphisms) $T_i : C^i \rightarrow C_1^i$ such that $f_i - g_i - d_{i-1} T_i = T_{i+1} d_i$ on $\text{Dom}(d_i)$ for all i . The homotopy is an equivalence relation on the set of all morphisms. Hence the notion of *chain-homotopy equivalence* of Hilbert cochain \mathcal{A} -complexes is well defined.

Definition 2.1. We will say that a Hilbert \mathcal{A} -complex C is *Fredholm* if there exists a Hilbert cochain \mathcal{A} -complex D of finite length such that the following two conditions are satisfied:

- (i) all D_i are finitely generated Hilbert \mathcal{A} -modules and all the differentials are bounded (i.e. D is a complex in the category $\mathcal{H}(\mathcal{A})$);
- (ii) D is chain-homotopy equivalent to C in the sense described above.

A pair (D, f) , consisting of a chain complex D of finite length in $\mathcal{H}(\mathcal{A})$ and of a chain-homotopy equivalence $f : D \rightarrow C$ will be called a *finite approximation* of C .

For a Fredholm Hilbert \mathcal{A} -complex C we may define its *extended cohomology* $\mathcal{H}^*(C)$ as the extended cohomology $\mathcal{H}^*(D)$, where (D, f) is an arbitrary finite approximation of C .

Suppose that we have a cochain map $g : C_1 \rightarrow C_2$ between two Fredholm Hilbert cochain \mathcal{A} -complexes. Let

$$f_1 : D_1 \rightarrow C_1 \quad \text{and} \quad f_2 : D_2 \rightarrow C_2$$

be finite approximations. Then the composition

$$D_1 \xrightarrow{f_1} C_1 \xrightarrow{g} C_2 \xrightarrow{h} D_2,$$

where h is a homotopy inverse to f_2 , is a chain map between two cochain complexes in $\mathcal{H}(\mathcal{A})$. Thus there is well-defined induced map $\mathcal{H}^i(D_1) \rightarrow \mathcal{H}^i(D_2)$ which we understand as the morphism induced by the original cochain map g ; sometimes we will denote this map $g^* : \mathcal{H}^i(C_1) \rightarrow \mathcal{H}^i(C_2)$.

In particular we see that there exists a canonical isomorphism $\mathcal{H}^i(D_1) \rightarrow \mathcal{H}^i(D_2)$ if D_1 and D_2 are two arbitrary finite approximations of a given Fredholm complex C . Thus, the extended cohomology of a finite approximation of a Fredholm complex does not depend on the approximation.

In the geometric situation, the complex C will arise as a de Rham complex of L^2 -differential forms on a compact smooth manifold (without boundary) with values in a flat Hilbert bundle over \mathcal{A} . We will give constructions of two finite approximations of this de Rham complex (constructions of this kind we will call de Rham theorems). In particular, we will establish that the finite cochain complex in $\mathcal{H}(\mathcal{A})$ constructed by using a cell decomposition of the manifold (which was used in [Fa] to define the extended cohomology) is chain-homotopy equivalent to the de Rham complex; this will be our version of the de Rham theorem for extended L^2 -cohomology.

3. DE RHAM COMPLEX

Let \mathcal{A} be a finite von Neumann algebra with a fixed trace τ , and E a fixed Hilbert \mathcal{A} -bundle over a compact closed manifold X . Let M be the standard fiber of E , so M is a fixed finitely generated Hilbert \mathcal{A} -module.

Let $\{g_{UV}\}$ be a 1-cocycle (with values in $GL_{\mathcal{A}}(M)$) which defines E .

For any open set $U \subset X$ denote by $\Lambda^p(U)$ the set of all smooth p -forms on U , and $\Lambda^\bullet(U) = \bigoplus_p \Lambda^p(U)$.

Denote by $C^\infty(U, M)$ the Fréchet space of smooth functions on U with values in M . The smooth p -forms on U with values in M form a Fréchet space

$$\Lambda^p(U, M) = \Lambda^p(U) \otimes_{C^\infty(U)} C^\infty(U, M).$$

The Fréchet space of smooth p -forms with values in E can be defined as

$$\Lambda^p(X, E) = \Lambda^p(X) \otimes_{C^\infty(X)} C^\infty(X, E),$$

where $C^\infty(X, E)$ is the space of C^∞ -sections of E over X . In other words a form $\omega \in \Lambda^p(X, E)$ is a collection $\omega = \{\omega_U\}_{U \in \mathcal{U}}$ where $\omega_U \in \Lambda^p(U, M)$, such that for any pair $U, V \in \mathcal{U}$ with $U \cap V \neq \emptyset$

$$\omega_U|_{U \cap V} = g_{UV} \cdot \omega_V|_{U \cap V}. \quad (3.1)$$

The exterior differential naturally extends to the operator

$$d = d^E : \Lambda^p(X, E) \longrightarrow \Lambda^{p+1}(X, E), \quad (3.2)$$

which is defined by applying the usual de Rham differential d to all ω_U (this is well-defined because the functions g_{UV} in the cocycle condition (3.1) are constant).

The operator d^E is also called the *covariant derivative* in E . Clearly $(d^E)^2 = 0$, so we have a well-defined de Rham complex

$$0 \longrightarrow \Lambda^0(X, E) \xrightarrow{d} \dots \xrightarrow{d} \Lambda^p(X, E) \xrightarrow{d} \Lambda^{p+1}(X, E) \xrightarrow{d} \dots \xrightarrow{d} \Lambda^n(X, E) \longrightarrow 0, \quad (3.3)$$

where $n = \dim_{\mathbb{R}} X$.

Let us choose a Riemannian metric on X and a smooth Hermitian metric on the bundle E , such that the topology it induces in each fiber coincides with the standard topology of that fiber. Then we can define the space $L^2\Lambda^p(X, E)$ of E -valued L^2 -forms of degree p . This space does not depend on the choices of the Riemannian metric on X and the Hermitian metric on E . Now we can define the L^2 -de Rham complex

$$0 \longrightarrow \dots \longrightarrow L^2\Lambda^p(X, E) \xrightarrow{d} L^2\Lambda^{p+1}(X, E) \longrightarrow \dots \longrightarrow 0, \quad (3.4)$$

where the differentials are closures of the corresponding differentials in (3.3). Clearly this complex is a Hilbert \mathcal{A} -complex in the sense described in Sect.2.

Now we can formulate our de Rham theorem. A similar theorem for smooth forms has been proved by M.Farber [Fa1].

Theorem 3.1. *Let X be a compact closed C^∞ -manifold without boundary. Then the de Rham complex (3.4) is Fredholm in the sense of Definition 2.1. Its extended cohomology coincides with the combinatorially defined extended cohomology of X with coefficients in E .*

In the next two sections we will give two constructions of finite approximations for the complex (3.4). In particular we will prove that this complex is homotopy equivalent to the combinatorial cochain L^2 -complex.

4. WITTEN APPROXIMATION

We shall use the same notations as in the previous section. In particular X will be a compact closed C^∞ -manifold.

Let $f : X \rightarrow \mathbb{R}$ be a Morse function on X . Consider the Witten deformations of the de Rham complexes (3.3) and (3.4) which are obtained by replacing d by the deformed differential

$$d_t = e^{-tf} de^{tf} = d + tdf \wedge \cdot, \quad t \in \mathbb{R}.$$

In this way we obtain new complexes

$$0 \rightarrow \dots \rightarrow \Lambda^p(X, E) \xrightarrow{d_t} \Lambda^{p+1}(X, E) \rightarrow \dots \rightarrow 0, \quad (4.1)$$

and

$$0 \rightarrow \dots \rightarrow L^2\Lambda^p(X, E) \xrightarrow{d_t} L^2\Lambda^{p+1}(X, E) \rightarrow \dots \rightarrow 0. \quad (4.2)$$

Assume now that a Riemannian metric g is chosen on X . Then we can also define the corresponding Witten Laplacian

$$\Delta_t = d_t^* d_t + d_t d_t^*. \quad (4.3)$$

It acts in $\Lambda^p(X, E)$ for every p and also defines a self-adjoint operator in $L^2\Lambda^p(X, E)$.

Following [BFKM] we will assume for simplicity that the pair (f, g) forms a *generalized triangulation*. This means that the following conditions are satisfied:

- (i) f is self-indexing i.e. $f(x) = \text{index}(x)$ for any critical point x of f ;
- (ii) in a neighborhood of any critical point x of f there exist coordinates y_1, \dots, y_n such that in these coordinates f has the form

$$f(y) = k - (y_1^2 + \dots + y_k^2)/2 + (y_{k+1}^2 + \dots + y_n^2)/2,$$

where $k = \text{index}(x)$;

- (iii) for any two critical points x and y of f , the unstable manifold W_x^- and the stable manifold W_y^+ , associated to the vector field $\text{grad}_g f$ intersect transversally.

Lemma 4.1. ([BFKM]) *There exist positive constants C', C'' and t_0 such that for any $t \geq t_0$*

$$\text{spec}(\Delta_t) \cap (e^{-tC'}, C''t) = \emptyset,$$

where spec means the spectrum in $L^2\Lambda^\bullet(X, E)$.

Remark. A stronger statement about the limit behavior of the whole spectrum was independently proved in [Sh2]. In [Sh2] only a particular case of the von Neumann algebra and the Hilbert module associated with a regular covering of X is considered but the arguments are easily extended to the general situation.

Lemma 4.1 allows to define a splitting of the deformed de Rham complex (4.2) into a direct sum of two Hilbert \mathcal{A} -subcomplexes:

$$L^2\Lambda^\bullet(X, E) = L^2\Lambda_{\text{sm}}^\bullet(X, E) \oplus L^2\Lambda_{\text{la}}^\bullet(X, E), \quad t > t_0. \quad (4.4)$$

Here $L^2\Lambda_{\text{sm}}^\bullet(X, E)$ and $L^2\Lambda_{\text{la}}^\bullet(X, E)$ are the images of the spectral projections of Δ_t corresponding to “small” and “large” eigenvalues (i.e. to the spectral intervals $[0, 1)$ and $[1, \infty)$, respectively). Denote these projections by $P_{\text{sm}}(t)$ and $P_{\text{la}}(t)$. They commute with the action of \mathcal{A} . They also commute with d_t because Δ_t commutes with d_t . Therefore both projections $P_{\text{sm}}(t)$ and $P_{\text{la}}(t)$ are morphisms of Hilbert cochain complexes.

Of course the decomposition (4.4) depends on t but we do not indicate this explicitly so as to simplify the notations.

All the complexes in (4.4) are considered with the differential d_t .

Denote by $i_{\text{sm}}(t) : L^2\Lambda_{\text{sm}}^\bullet(X, E) \rightarrow L^2\Lambda^\bullet(X, E)$ the canonical inclusion. It is also a morphism of Hilbert cochain \mathcal{A} -complexes.

Theorem 4.2. (i) *The maps $i_{\text{sm}}(t)$ and $P_{\text{sm}}(t)$ define a chain-homotopy equivalence of the Hilbert cochain \mathcal{A} -complexes $L^2\Lambda_{\text{sm}}^\bullet(X, E)$ and $L^2\Lambda^\bullet(X, E)$ (with the differentials d_t) for any $t > t_0$.*

(ii) *The differentials d_t are bounded \mathcal{A} -operators in $L^2\Lambda_{\text{sm}}^\bullet(X, E)$.*

(iii) *$L^2\Lambda_{\text{sm}}^\bullet(X, E)$ is a finitely generated Hilbert cochain \mathcal{A} -complex.*

Corollary 4.3. *The de Rham complex $L^2\Lambda_{\text{sm}}^\bullet(X, E)$ is a Fredholm Hilbert cochain \mathcal{A} -complex in the sense of Definition 2.1.*

So Theorem 4.2. gives us the first finite approximation of the de Rham complex.

Proof of Theorem 4.2. We have

$$\|d_t\omega\|^2 = (d_t^*d_t\omega, \omega) \leq (\Delta_t\omega, \omega) \leq \|\omega\|^2, \quad \omega \in L^2\Lambda_{\text{sm}}^\bullet(X, E),$$

which proves (ii).

To prove (i) let us introduce a Green operator G_t by the formula

$$G_t = \Delta_t^{-1}(I - P_{\text{sm}}(t)) : L^2\Lambda^\bullet(X, E) \rightarrow L^2\Lambda^\bullet(X, E),$$

where I is the identity operator, Δ_t^{-1} is the operator which is inverse to Δ_t restricted to $(L^2\Lambda_{\text{sm}}^\bullet(X, E))^\perp = \text{Im}(I - P_{\text{sm}}(t))$, and by definition $G_t = 0$ on $L^2\Lambda_{\text{sm}}^\bullet(X, E) = \text{Im} P_{\text{sm}}(t)$. Clearly G_t is a bounded \mathcal{A} -operator. It follows easily from elliptic regularity arguments that G_t is smoothing by 2 units in the corresponding Sobolev scale (see e.g. similar arguments in [Sh]). In particular, the operators $d_t G_t$, $d_t^* G_t$, $d_t^* d_t G_t$ and $d_t d_t^* G_t$ are bounded in $L^2\Lambda^\bullet(X, E)$.

Note also that G_t commutes with d_t and d_t^* . Therefore we obtain

$$I - P_{\text{sm}}(t) = \Delta_t \Delta_t^{-1} (I - P_{\text{sm}}(t)) = d_t d_t^* G_t + d_t^* d_t G_t = d_t (d_t^* G_t) + (d_t^* G_t) d_t,$$

so the operator $d_t^* G_t$ supplies a chain homotopy between the \mathcal{A} -endomorphisms I and $P_{\text{sm}}(t)$ of the Hilbert cochain \mathcal{A} -complex $L^2\Lambda^\bullet(X, E)$. This proves (i).

To prove (iii) we use Theorem 5.5 from [BFKM] which provides an isomorphism of Hilbert cochain \mathcal{A} -complexes

$$\phi : (L^2\Lambda^\bullet(X, E), \tilde{d}_t) \longrightarrow (\mathcal{C}^\bullet, \delta), \quad (4.5)$$

where \tilde{d}_t is obtained from d_t by scaling

$$\tilde{d}_t = e^t \left(\frac{t}{\pi} \right)^{-1/2} d_t,$$

and $(\mathcal{C}^\bullet, \delta)$ is a finite combinatorial \mathcal{A} -complex associated with the given bundle E and the chosen generalized triangulation (see Section 4 in [BFKM] and also [BZ1] for the case $\mathcal{A} = \mathbb{C}$). Since the complex $(\mathcal{C}^\bullet, \delta)$ is obviously finitely generated, we obtain (iii). \square

Corollary 4.4. *The extended de Rham cohomology with coefficients in E coincides with extended combinatorial cohomology with the same coefficients.*

Proof. The L^2 -de Rham complex $(L^2\Lambda^\bullet(X, E), d)$ is obviously isomorphic to its Witten deformation $(L^2\Lambda^\bullet(X, E), d_t)$, so they have the same extended cohomology. On the other hand, replacing d_t by \tilde{d}_t also leads to an isomorphic complex. It

follows that the combinatorial complex $(\mathcal{C}^\bullet, \delta)$ is a finite approximation for the de Rham complex, therefore it can be used to calculate its extended cohomology. \square

Proof of Theorem 3.1. According to Theorem 6.4 in [Fa] we can use any CW-complex structure on X to calculate the extended cohomology. In particular we can use the above-mentioned complex $(\mathcal{C}^\bullet, \delta)$. \square

Remark. We see that both $(\mathcal{C}^\bullet, \delta)$ and $(L^2\Lambda_{sm}^\bullet(X, E), d_t)$ provide finite approximations of the de Rham complex $(L^2\Lambda^\bullet(X, E), d)$. Another finite approximation will be given in the next section.

5. SPECTRAL APPROXIMATION

In this section we will show that it is possible to construct a finite approximation of the de Rham \mathcal{A} -complex (3.4) by the direct spectral cut-off i.e. without first deforming as above.

As before let X be a closed smooth manifold and E a flat Hilbert \mathcal{A} -bundle over X . Choose a Riemannian metric on X and a Hermitian metric on E . Then the Hilbert spaces of E -valued forms $L^2\Lambda^p(X, E)$ and $L^2\Lambda^\bullet(X, E)$ are well defined. We can also consider the Laplacian

$$\Delta^{(p)} = d^*d + dd^* : \Lambda^p(X, E) \longrightarrow \Lambda^p(X, E)$$

which is a self-adjoint operator in $L^2\Lambda^p(X, E)$. Denote also $\Delta = \bigoplus_p \Delta^{(p)}$, so Δ acts in $\Lambda^\bullet(X, E)$ and is a self-adjoint operator in $L^2\Lambda^\bullet(X, E)$.

Denote by $E_\lambda^{(p)}$ the spectral projection of the Laplacian $\Delta^{(p)}$ corresponding to the spectral interval $[0, \lambda]$ where $\lambda > 0$. Denote $L_\lambda^{(p)} = \text{Im } E_\lambda^{(p)}$, so $L_\lambda^{(p)}$ is a closed subspace in $L^2\Lambda^p(X, E)$.

Theorem 5.1.

(i) $L_\lambda^{(p)} \subset \Lambda^p(X, E)$ (i.e. $L_\lambda^{(p)}$ consists of smooth E -valued forms) and $L_\lambda^{(p)}$ is a Hilbert \mathcal{A} -submodule in $L^2\Lambda^p(X, E)$.

(ii) $d(L_\lambda^{(p)}) \subset L_\lambda^{(p+1)}$ and $d : L_\lambda^{(p)} \rightarrow L_\lambda^{(p+1)}$ is a bounded \mathcal{A} -operator.

(iii) The inclusion $i_\lambda : (L_\lambda^{(\bullet)}, d) \rightarrow (L^{(\bullet)}, d)$ is a chain-homotopy equivalence in the sense of section 2; the homotopy inverse map is given by the projection $E_\lambda^{(p)}$.

(iv) The Hilbertian \mathcal{A} -module $L_\lambda^{(p)}$ is finitely generated for every p and every $\lambda > 0$.

Thus the complex $(L_\lambda^{(\bullet)}, d)$ corresponding to “small eigenvalues” provides another finite approximation of the de Rham complex.

Proof of (i)-(iii) in Theorem 5.1. The proof is done by the same arguments as the proof of (i) and (ii) in Theorem 4.2. Namely, (i) follows from elliptic regularity, the proof of (ii) is identical to the proof of (ii) in Theorem 4.2, and the proof of (iii) uses a homotopy operator constructed from a Green operator as in the proof of (i) in Theorem 4.2. \square

To prove (iv) we will first consider the simplest case.

Proof of (iv) when \mathcal{A} is a factor. We have to prove that $L_\lambda^{(p)}$ is finitely generated. We shall use the natural extension of the trace τ to morphisms of Hilbert \mathcal{A} -modules. By an abuse of notation we will also denote this extension by τ . Note first that $\tau(E_\lambda^{(p)}) < \infty$ due to arguments similar to the ones given in [A].

The assumption that \mathcal{A} is a factor allows us to use the following simplifying fact: for any two Hilbert \mathcal{A} -modules L_1, L_2 a continuous inclusion of \mathcal{A} -modules $L_1 \subset L_2$ exists if and only if $\dim_\tau L_1 \leq \dim_\tau L_2$ where \dim_τ is the von Neumann dimension function induced by the trace τ . Since $\dim_\tau L_\lambda^{(p)} = \tau(E_\lambda^{(p)}) < \infty$, the Hilbert \mathcal{A} -module $L_\lambda^{(p)}$ can be embedded as a Hilbert \mathcal{A} -submodule into a free Hilbert \mathcal{A} -module $\ell^2(\mathcal{A}) \otimes \mathbb{C}^N$, for a sufficiently large integer N . This means by definition that $L_\lambda^{(p)}$ is finitely generated. \square

The general case requires some preparations concerning direct integral factor decompositions of the algebra \mathcal{A} and finitely generated Hilbert \mathcal{A} -modules.

In the general case \mathcal{A} is a direct integral of finite factors. We shall deduce the proof in this case from the case when \mathcal{A} is a factor. Let us start with a description of the finitely generated Hilbert \mathcal{A} -modules in terms of their direct integral factor decomposition.

We refer to [Di,P,Sc] for results about the direct integral decompositions of von Neumann algebras and traces.

Consider the direct integral factor decomposition of \mathcal{A} :

$$\mathcal{A} = \int_{\Omega} \oplus \mathcal{A}(\omega) \mu(d\omega) \quad (5.1)$$

Here Ω is a Borel space, $\omega \mapsto \mathcal{A}(\omega)$ is a Borel family of finite factors, $\mu(d\omega)$ is a finite Borel measure on Ω . We will choose the Borel space Ω and the measure $\mu(d\omega)$ in a special way: we will assume that \mathcal{A} has a direct integral decomposition (5.1) in its regular representation, i.e. in its natural representation in $\ell^2(\mathcal{A})$. So we have

$$\ell^2(\mathcal{A}) = \int_{\Omega} \oplus h(\omega) \mu(d\omega), \quad (5.2)$$

where $h(\omega)$ is a Hilbert space where $\mathcal{A}(\omega)$ is represented.

Now let us consider an arbitrary finitely generated Hilbert \mathcal{A} -module M .

Lemma 5.2. *M has a direct integral decomposition*

$$M = \int_{\Omega} M(\omega) \mu(d\omega), \quad (5.3)$$

where $M(\omega)$ is a Hilbert $\mathcal{A}(\omega)$ -module and in this decomposition the action of \mathcal{A} is diagonal in the obvious sense with respect to the decomposition (5.1).

Proof. Let M be a Hilbert submodule in $(\ell^2(\mathcal{A}))^N = \ell^2(\mathcal{A}) \otimes \mathbb{C}^N$, and P be the orthogonal projection in $(\ell^2(\mathcal{A}))^N$ with $\text{Im } P = M$. Clearly P should be in the commutant of the diagonal action of \mathcal{A} in $(\ell^2(\mathcal{A}))^N$. According to Theorem 4.11.8 in [P] this commutant is again the direct integral

$$\mathcal{A}' = \int_{\Omega} \oplus \mathcal{A}'(\omega) \mu(d\omega) \quad (5.4)$$

with the diagonal action in $(\ell^2(\mathcal{A}))^N$. Therefore we have

$$P = \int_{\Omega} \oplus P(\omega) \mu(d\omega), \quad (5.5)$$

where $P(\omega)$ is an orthogonal projection in $h(\omega)$ such that $P(\omega) \in \mathcal{A}'(\omega)$. Denote $M(\omega) = \text{Im } P(\omega) \subset h(\omega) \otimes \mathbb{C}^N$. Then (5.5) implies the decomposition (5.2). \square

The trace τ on \mathcal{A} can be written in the form

$$\tau(a) = \int_{\Omega} \tau_{\omega}(a(\omega)) \mu(d\omega), \quad a \in \mathcal{A}, \quad (5.6)$$

where $a = (a(\omega))$ in the direct integral decomposition (5.1), and τ_{ω} is a trace on $\mathcal{A}(\omega)$.

Note that we do not assume the traces τ_{ω} to be normalized. For future use denote $\rho(\omega) = \tau_{\omega}(I_{\mathcal{A}(\omega)})$. Then $\tau_{\omega} = \rho(\omega) \bar{\tau}_{\omega}$ where $\bar{\tau}_{\omega}$ is the normalized trace on the factor $\mathcal{A}(\omega)$. Clearly $\rho \in L^1(\omega, \mu)$, $\rho \geq 0$. The formula (5.6) can be rewritten as follows:

$$\tau(a) = \int_{\Omega} \rho(\omega) \bar{\tau}_{\omega}(a(\omega)) \mu(d\omega), \quad a \in \mathcal{A}. \quad (5.7)$$

The trace τ is normalized if and only if $\int_{\Omega} \rho(\omega) \mu(d\omega) = 1$.

Lemma 5.3. *M is finitely generated if and only if*

$$\text{ess sup}_{\omega \in \Omega} (\rho(\omega)^{-1} \dim_{\tau(\omega)} M(\omega)) < \infty. \quad (5.8)$$

Proof. We refer to Theorem 11 in Chapter III of [Sc] for the following fact. Assume that

$$M = \int_{\Omega} M(\omega) \mu(d\omega), \quad M_1 = \int_{\Omega} M_1(\omega) \mu(d\omega), \quad (5.9)$$

are two Hilbert \mathcal{A} -modules decomposed into direct integrals according to the decomposition (5.1) of the algebra \mathcal{A} . A continuous linear imbedding of \mathcal{A} -modules $M \subset M_1$ exists iff

$$\dim_{\tau(\omega)} M(\omega) \leq \dim_{\tau(\omega)} M_1(\omega) \quad (5.10)$$

for almost all ω . Applying this to compare the decomposition of M with the decomposition of $l^2(\mathcal{A}) \otimes \mathbb{C}^N$ we obtain the statement of the Lemma. \square

Proof of (iv) in the general case. Note first that it follows from the decomposition of the commutant (5.4) that any representation of a discrete countable group Γ in M by automorphisms of M as a Hilbert \mathcal{A} -module can be decomposed

into a direct integral of actions of Γ in $M(\omega)$. This leads to a decomposition of any flat Hilbert bundle with fiber M over a compact manifold X into a direct integral of flat Hilbert bundles with fibers $M(\omega)$. This leads to the decomposition of the space of L^2 -forms with values in the Hilbert bundle, and furthermore to the decomposition of the Laplacian $\Delta^{(p)}$ into a direct integral of Laplacians $\Delta^{(p)}(\omega)$ operating on forms with values in flat vector bundles $E(\omega)$ with fibers $M(\omega)$. The cocycle defining the bundle $E(\omega)$ is obtained from the components $g_{UV}(\omega)$ in the direct decomposition of the cocycle g_{UV} defining the bundle E .

All the functions of the Laplacian $\Delta^{(p)}$ are then also decomposed. In particular, for the spectral projection $E_\lambda^{(p)}$ we have the decomposition

$$E_\lambda^{(p)} = \int_{\Omega} E_\lambda^{(p)}(\omega) d\mu(\omega), \quad (5.11)$$

where $E_\lambda^{(p)}(\omega)$ is the spectral projection of $\Delta^{(p)}(\omega)$. Taking into account Lemma 5.3 we see that the statement (iv) of Theorem 5.1 will be proved if we establish the estimate

$$\text{ess sup}_{\omega \in \Omega} (\rho(\omega)^{-1} \tau_\omega(E_\lambda^{(p)}(\omega))) < \infty \quad (5.12)$$

for any $\lambda > 0$.

To prove this note first that for the Schwartz kernel $E_\lambda^{(p)}(\omega)(x, y)$ of the spectral projection $E_\lambda^{(j)}(\omega)$ we have an estimate

$$\text{ess sup}_{\omega \in \Omega} \sup_{x, y \in X} \|E_\lambda^{(j)}(\omega)(x, y)\| \leq C < \infty. \quad (5.13)$$

This follows from standard elliptic estimates (see e.g. arguments given in [Sh, FS, CG], for similar estimates). The constant C in (5.13) does not depend on ω (but may depend on λ and on all the other data, e.g. X, \mathcal{A} and E).

Indeed the L^2 -operator norm of the orthogonal projection equals 1. Using the elliptic estimates we can now prove that in fact the norm of the projection is bounded not only in L^2 but also in the Sobolev spaces. More precisely the norm of $E_\lambda^{(p)}$ as an operator from H^{-s} to H^s for any $s > 0$ is bounded, so that the operator is infinitely smoothing. The constants in these estimates depend on bounds for the coefficients, their derivatives and the inverted symbol (the last is to insure the uniform ellipticity). Therefore these estimates are uniform with respect to ω . Using them together with the Sobolev embedding theorem ($H^s \subset C$ for $s > \dim X/2$ which implies that the Dirac delta-function belongs to H^{-s} for $s > \dim X/2$) we arrive to (5.13).

Now note that due to Lemma 5.3

$$\text{ess sup}_{\omega \in \Omega} (\rho(\omega)^{-1} \tau_\omega(I_\omega)) = \text{ess sup}_{\omega \in \Omega} (\rho(\omega)^{-1} \dim_{\tau_\omega} M(\omega)) < \infty$$

because the fiber M is finitely generated. (Here I_ω is the identity operator in $M(\omega)$.)

Using the estimate

$$|\tau_\omega(T)| \leq \|T\| \tau_\omega(I_\omega),$$

which is true for any endomorphism T of the Hilbert module $M(\omega)$, we immediately arrive at the estimate (5.2). This ends the proof of (iv) in Theorem 5.1. \square

Remark. Let \mathcal{A} be a finite factor and M a Hilbert \mathcal{A} -module. Then M is finitely generated if and only if $\dim_{\tau} M < \infty$. But this is not true for general finite von Neumann algebras. The following example is due to M.Farber. Consider $\mathcal{A} = L^{\infty}([0, 1])$ (where $[0, 1]$ is taken with the Lebesgue measure) with the trace τ given by the Lebesgue integral, and take

$$M = L^2([0, 1/2]) \oplus (L^2([1/2, 3/4]) \otimes \mathbb{C}^2) \oplus (L^2([3/4, 7/8]) \otimes \mathbb{C}^3) \oplus \dots$$

Here $a \in L^{\infty}([0, 1])$ acts in each space $L^2([(2^{k-1} - 1)/2^{k-1}, (2^k - 1)/2^k]) \otimes \mathbb{C}^k$ as $a_k \otimes I_k$ where a_k is the restriction of a to $[(2^{k-1} - 1)/2^{k-1}, (2^k - 1)/2^k]$ and I_k is the identity operator in \mathbb{C}^k . Clearly

$$\dim_{\tau} M = \sum_{k=1}^{\infty} \frac{k}{2^k} < \infty,$$

but M is not finitely generated.

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DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115
E-mail address: `shubin@neu.edu`