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THE FAVARD-MUHAMADIEV THEORY AND PSEUDODIFFERENTIAL OPERATORS

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In this note it is established that, for an elliptic differential operator with smooth almost periodic (a.p.) coefficients, the classical Favard condition is equivalent to the existence of an inverse operator that is an almost periodic pseudodifferential operator of a certain type.

1. Let A be a differential operator in $\mathbf{R}^n$ of the form

$$A=\sum_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha, \tag{1}$$

where $x \in \mathbf{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, the α_j are nonnegative integers, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $D_j = i^{-1}\,\partial/\partial x_j$. We will assume that the following conditions are satisfied:

a) the $a_\alpha(x)$ are uniformly a.p. functions on $\mathbf{R}^n$;

b) $|D^\beta a_\alpha(x)| \le C_{\alpha\beta}$ for any multi-index β and for each α with $|\alpha| \le m$;

c) the operator (1) is uniformly elliptic, i.e. there exists an $\epsilon > 0$ such that

$$\Big|\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha\Big| \geqslant \varepsilon|\xi|^m, \qquad \xi\in\mathbf{R}^n. \tag{2}$$

We note that it follows from conditions a) and b) that all of the derivatives $D^\beta a_\alpha(z)$ are also uniformly a.p. functions on $\mathbf{R}^n$.

We introduce the hull $H(A)$ of the operator A, consisting of the operators $\tilde{A} = \sum_{|\alpha|\le m}\tilde{a}_\alpha(x)D^\alpha$ for which there exists a sequence $\{h_k\}$, $k = 1, 2, \ldots$, of vectors in $\mathbf{R}^n$ such that

$$\lim_{k\to+\infty}\ \sup_{x\in\mathbf{R}^n} |a_\alpha(x+h_k)-\tilde{a}_\alpha(x)|=0, \qquad |\alpha|\leqslant m. \tag{3}$$

We note that (3) implies, in view of condition b), the uniform convergence of any of the derivatives $D^\beta a_\alpha(x + h_k)$ as $k \to +\infty$, which in turn implies that any operator $\tilde{A} \in H(A)$ satisfies conditions a), b), c).

We now formulate Favard's condition:

F) for any $\tilde{A} \in H(A)$, the equation $\tilde{A}u = 0$ has no bounded solutions.

This condition first appeared in a classical paper of Favard [2] in the case $n = 1$, i.e. for ordinary differential equations. Favard proved that this condition guarantees the uniform almost periodicity of any bounded solution $u(x)$ of the equation $Au = f$, where f is a uniformly a.p. function. An important addition was made to the Favard

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theory by Muhamadiev [3], who proved in the same situation the solvability in the class of bounded functions of the equation $Au = f$. Later, Muhamadiev extended the Favard theory and the solvability theorem to the case of arbitrary n (see [4]), i.e. to the case of partial differential equations. In [4] he proved, in particular, that condition F) implies the invertibility of $A: C_{m+\gamma} \to C_\gamma$, where $0 < \gamma < 1$, C_γ is the space of bounded functions on $\mathbf{R}^n$ satisfying a Hölder condition of order γ, and $C_{m+\gamma}$ is the space of bounded functions on $\mathbf{R}^n$ all of whose derivatives of order $\leq m$ are bounded and whose derivatives of order m belong to C_γ.

We introduce here, in addition, the space $C_b^\infty(\mathbf{R}^n)$ consisting of the functions $f(x) \in C^\infty(\mathbf{R}^n)$ all of whose derivatives are bounded, i.e. $|D^\alpha f(x)| \leq C_\alpha$ for any multi-index α (the constants C_α, of course, depend on f). We note that every bounded solution u of the equation $\tilde{A}u = 0$ belongs to $C_b^\infty(\mathbf{R}^n)$ by virtue of the standard a priori estimate.

2. We will use the classes $APL^m_{\rho,\delta}$, $APL^{-\infty}$ of pseudodifferential operators and the corresponding classes $APS^m_{\rho,\delta}$, $APS^{-\infty}$ of symbols introduced in [5]. The relation $a(x, \xi) \in APS^m_{\rho,\delta}$ means, first, that $\partial_\xi^\alpha \partial_x^\beta a(\cdot, \xi)$ is for any multi-indices α, β a continuous function of $\xi \in \mathbf{R}^n$ with values in the space $CAP(\mathbf{R}^n_x)$ of uniformly a.p. functions of the n-dimensional variable x and, second, that the estimates

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leqslant C_{\alpha\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|} \tag{4}$$

are satisfied. Here, as usual, we will assume that $0 \leq \delta < 1$ and $0 < \rho \leq 1$. Also, by definition, $APS^{-\infty} = \bigcap_m APS^m_{1,0}$. An operator $A = Op(a)$ with symbol $a(x, \xi)$ is defined by the oscillating integral

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(x, \xi) u(y)\, dy\, d\xi, \tag{5}$$

where $x \cdot \xi$ denotes the usual scalar product in $\mathbf{R}^n$, and dy and $d\xi$ are the Lebesgue measures in the corresponding n-dimensional spaces. By $APL^m_{\rho,\delta}$ and $APL^{-\infty}$ are denoted the classes of operators of the form $A = Op(a)$ with $a \in APS^m_{\rho,\delta}$ and $a \in APS^{-\infty}$ respectively. An operator $A \in APL^m_{\rho,\delta}$ maps $S(\mathbf{R}^n)$ into $S(\mathbf{R}^n)$ and $C_b^\infty(\mathbf{R}^n)$ into $C_b^\infty(\mathbf{R}^n)$ (see, for example, [5]).

We introduce, in addition, the more restricted class APL^m_{cl} of classical a.p. pseudodifferential operators whose symbols $a(x, \xi) \in APS^m_{1,0}$ admit an asymptotic expansion in functions $a_{m-j}(x, \xi)$, $j = 0, 1, \ldots$, that are homogeneous in ξ of degree $m - j$, i.e.

$$a_{m-j}(x, t\xi) = t^{m-j} a_{m-j}(x, \xi), \quad t > 0, \quad |\xi| \neq 0. \tag{6}$$

More precisely, we will write $A \in APL^m_{\mathrm{cl}}$ if $A = Op(a)$ with $a(x, \xi) \in C^\infty(\mathbf{R}^{2n})$ and there exists a collection of functions $a_{m-j}(x, \xi) \in C^\infty(\mathbf{R}^n \times (\mathbf{R}^n \setminus 0))$ satisfying condition (6) such that (i) $\partial_\xi^\alpha \partial_x^\beta a_{m-j}(\cdot, \xi)$ is for any multi-indices α and β a continuous function of ξ with values in $CAP(\mathbf{R}^n)$ and (ii) if $\theta(\xi) \in C^\infty(\mathbf{R}^n)$, $\theta(\xi) = 0$ for $|\xi| \leq 1/2$ and $\theta(\xi) = 1$ for $|\xi| \geq 1$, then

$$a(x, \xi) - \sum_{j=0}^{N-1} \theta(\xi) a_{m-j}(x, \xi) \in APS^{m-N}_{1,0}$$

for any natural number N.

We can now formulate the main result.

Theorem 1. *Suppose A is an operator of the form* (1) *satisfying conditions* a), b), c). *Then*

1) *there exists an inverse $A^{-1} \in APL_{\mathrm{cl}}^{-m}$ of A if condition* F) *is satisfied;*

2) *conversely, condition* F) *is satisfied if there exists an inverse $A^{-1} \in APL_{\rho,\delta}^{M}$ of A (M is any real number, $0 < \rho \leq 1$, $0 \leq \delta < 1$).*

Remark 1. The fact that the operators A and A^{-1} are the inverses of each other is most conveniently understood and verified in the spaces $S(\mathbf{R}^n)$ and $C_b^{\infty}(\mathbf{R}^n)$, which are mapped by these operators into themselves. Closure then permits one to obtain the inverse of A in other spaces. For example, it is possible in this way to prove the invertibility of $A\colon H_s(\mathbf{R}^n) \to H_{s-m}(\mathbf{R}^n)$ as well as the invertibility of A in a series of other spaces (see, for example, the scales of spaces considered by Muhamadiev [4]).

Remark 2. Theorem 1 remains valid for systems of the form (1) if the $a_\alpha(x)$ are $N \times N$ matrices. In this connection, it is necessary that condition (2) be replaced by the condition of uniform ellipticity in the Petrovskiĭ sense and that one of the following three conditions be satisfied:

1) $a_\alpha(x) = a_\alpha = \text{const}$ for $|\alpha| = m$;

2) n is odd;

3) there exists a real number ϕ_0 such that $\arg \lambda_j(x, \xi) \neq \phi_0 \pmod{2\pi}$ for all of the eigenvalues $\lambda_j(x, \xi)$ of the matrix $a_m(x, \xi) = \sum_{|\alpha|=m} \xi^\alpha a_\alpha(x)$.

This requirement was introduced by Muhamadiev [4] and guarantees the absence of topological obstructions to invertibility that automatically vanish in the scalar case.

3. Proof of Theorem 1. We will make use of the fact that if conditions a), b), c) are satisfied, then A has a parametrix, viz. an operator $B \in APL_{\mathrm{cl}}^{-m}$ such that

$$BA = I + T_1, \qquad AB = I + T_2, \tag{7}$$

where $T_j \in APL^{-\infty}$, $j = 1, 2$ (see [5], [6]).

Let us now prove assertion 1) of the theorem. Suppose condition F) is satisfied. We will avail ourselves of a result of Muhamadiev [4] on the existence of a bounded inverse $A^{-1}\colon C_\gamma \to C_{m+\gamma}$ of $A\colon C_{m+\gamma} \to C_\gamma$. It follows from Theorem 2.2 of [6] that the relation $A^{-1} - B \in APL^{-\infty}$ can be proved by establishing that A^{-1} maps $CAP(\mathbf{R}^n) \cap C_b^{\infty}(\mathbf{R}^n)$ into $CAP(\mathbf{R}^n)$. But this obviously follows from the just-mentioned result of Muhamadiev.

Let us prove assertion 2) of the theorem. We note that if the operator $B = A^{-1} \in APL_{\rho,\delta}^{M}$ has the symbol $b(x, \xi)$ (i.e. $B = Op(b)$), then each of the operators $\tilde{A}$ with coefficients $\tilde{a}_\alpha(x)$ given by formulas (3) also has an inverse $\tilde{B} \in APL_{\rho,\delta}^{M}$ with symbol

$$\tilde{b}(x, \xi) = \lim_{k \to +\infty} b(x + h'_k, \xi),$$

where $\{h'_k\}$ is a subsequence of the sequence $\{h_k\}$ in (3). Taking into account the fact that $\tilde{B}$ maps $C_b^{\infty}(\mathbf{R}^n)$ into $C_b^{\infty}(\mathbf{R}^n)$, we at once obtain the validity of condition F). Theorem 1 is proved.

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BIBLIOGRAPHY

1. B. M. Levitan, *Almost-periodic functions*, GITTL, Moscow, 1953. (Russian) MR 15, 700.
2. J. Favard, Acta Math. 51 (1927), 31.

3. È. Muhamadiev, Dokl. Akad. Nauk SSSR **196** (1971), 47 = Soviet Math. Dokl. **12** (1971), 49. MR 43 #2566.

4. È. Muhamadiev, Dokl. Akad. Nauk SSSR **205** (1972), 1292 = Soviet Math. Dokl. **13** (1972), 1122. MR 47 #2437.

5. M. A. Šubin, Mat. Sb. **95** (137) (1974), 560 = Math. USSR Sb. **24** (1974), 547.

6. ————, Trudy Moskov. Mat. Obšč. **35** (1975) = Trans. Moscow Math. Soc. **35** (1975) (to appear).

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