

ALGEBRAS OF PSEUDODIFFERENTIAL OPERATORS ON UNIMODULAR LIE GROUPS

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In [1] Hörmaner introduced a class of pseudodifferential operators (Ψ DO's) in \mathbf{R}^n whose symbols satisfy estimates that are uniform in the spatial variable. The algebra of such operators was subsequently studied by Kumano-go [2], [3], and by a number of other authors. In this note we construct analogous algebras of operators on an arbitrary unimodular Lie group. Here we have to impose conditions not only on the local symbols but also on the operator kernels, which must decrease off the diagonal more rapidly than any power of a weight function of the distance, where the weight function must have intermediate growth between the volume function and an exponential (in \mathbf{R}^n such a condition was not written previously in explicit form, because for a power weight function it follows from standard estimates of the symbols, and the case of more general weight functions was not considered). For uniformly elliptic operators we give estimates of the Green's function, and we construct complex powers of operators of the type under consideration. We note also that analogous algebras of operators on discrete groups were constructed in [5], and uniformly properly supported Φ DO's on Lie groups and Sobolev spaces on Lie groups were studied in [6].

1. Let G be a connected unimodular Lie group of dimension n , \mathfrak{G} its Lie algebra, dg Haar measure on G , and ρ the left-invariant metric on G induced by a left-invariant Riemannian metric. Let $B(g, r) = \{h: \rho(g, h) \leq r\}$ be the ball of radius r about a point g , and $V(r)$ the volume of this ball. We choose a basis X_1, \dots, X_n in \mathfrak{G} and regard its elements as left-invariant vector fields on G . For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_j \in \mathbf{Z}$ and $\alpha_j \geq 0$, define $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, so that X^α is a left-invariant differential operator on G of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

We introduce a positive nondecreasing function $\psi: [0, +\infty) \rightarrow (0, +\infty)$, having the following properties: there exist positive N and C such that

$$\psi(r+s) \leq C\psi^N(r)\psi^N(s), \quad r, s \geq 0, \quad V(r) \leq C\psi^N(r), \quad r \geq 0.$$

Since $V(r)$ increases no more rapidly than an exponential as $(r \rightarrow \infty V(r) \leq C^{ar}, r \geq 0)$, we can take $\psi(r) = e^r$ as ψ . On the other hand, on nilpotent groups, where $V(r)$ has power growth, we can also take $\psi(r) = 1 + r$ or $\psi(r) = e^{r^\alpha}$ with $0 < \alpha \leq 1$.

We shall use the usual Hörmander classes of symbols S^m and Ψ DO's L^m (see, for example, [7]). Choose some coordinate neighborhood U_e of the identity $e \in G$ with coordinates x_1, \dots, x_n regarded as defined in a neighborhood of the closure of U_e and as fixed throughout the whole article. In the neighborhood $U_g = gU_e$ of a point g we consider the coordinates $x_1^{(g)}, \dots, x_n^{(g)}$, obtained from x_1, \dots, x_n by left translation by g : $x_j^{(g)} = L_g^{-1}x_j$, where $L_g u(h) = u(gh)$.

DEFINITION 1. The class $UL^m(G, \psi)$ consists of the Ψ DO's $A \in L^m$ on G such that the following conditions hold:

a) The family $\{A_g: C_0^\infty(U_g) \rightarrow C^\infty(U_g)\}$ of restrictions of A to U_g forms a family of operators in $L^m(U_g)$ that is uniformly (with respect to g) bounded in the coordinates $x_1^{(g)}, \dots, x_n^{(g)}$.

b) The kernel $K_A(g, h)$ of A satisfies the estimates

$$|X_g^\alpha X_h^\beta K_A(g, h)| \leq C_{\alpha\beta} [\psi(\rho(g, h))]^{-M}, \quad \rho(g, h) \geq \varepsilon > 0,$$

where α and β are arbitrary multi-indices, ε and M are arbitrary positive constants, and the lower index g in the notation X_g^α means that we are considering the operator X^α applied with respect to the variable g .

Condition a) means that in the coordinates $x_1^{(g)}, \dots, x_n^{(g)}$ the operator A_g can be written as a sum $A_g = a_g(x, D_x) + R_g$, where $a_g(x, \xi) \in S^m$, with the constants in the estimates for the symbol a_g chosen independent of g , and R_g is an operator with smooth kernel whose derivatives (in the same coordinates) are all estimated by constants independent of g .

EXAMPLE. A differential operator $A = \sum_{|\alpha| \leq m} a_\alpha(g) X^\alpha$ belongs to $UL^m(G, \psi)$ if and only if $|X^\beta a_\alpha(g)| \leq C_{\alpha\beta}$ for any multi-indices α and β with $|\alpha| \leq m$.

It is easy to see that the class $UL^{-\infty}(G, \psi) = \bigcap_{m \in \mathbf{R}} UL^m(G, \psi)$ consists of the operators R with kernel $K_R \in C^\infty(G \times G)$ satisfying the estimates

$$|X_g^\alpha X_h^\beta K_R(g, h)| \leq C_{\alpha\beta} [\psi(\rho(g, h))]^{-M}.$$

On G we introduce the Schwartz space $S(G, \psi)$ with weight ψ consisting of the functions $f \in G^\infty(G)$ with finite seminorms

$$\|f\|_k = \sup_{\substack{g \in G \\ |\alpha| \leq k}} [\psi(\rho(e, g))]^k |X^\alpha f(g)|, \quad k = 0, 1, 2, \dots,$$

which determine a Fréchet space structure on $S(G, \psi)$. An operator $A \in L^m(G, \psi)$ gives a continuous mapping of $S(G, \psi)$ into itself. Further, let $L_2(G)$ be the usual space of square-integrable (with respect to Haar measure) functions on G .

THEOREM 1. a) If $A \in UL^{m_1}(G, \psi)$ and $B \in UL^{m_2}(G, \psi)$, then

$$AB \in UL^{m_1+m_2}(G, \psi).$$

b) If $A \in UL^m(G, \psi)$, then $A^* \in UL^m(G, \psi)$ (here A^* is the operator formally adjoint to A acting in $L^2(G)$).

c) If $A \in UL^0(G, \psi)$, then A extends to a bounded operator on $L_2(G)$.

We single out in $UL^m(G, \psi)$ the subclass $UL^m(G)$ of uniformly properly supported Ψ DO's, i.e., operators $A \in UL^m(G, \psi)$ with kernel K_A such that $K_A(g, h) = 0$ for $\rho(g, h) > C_A$. Then $UL^m(G)$ is a *-subalgebra of $UL^m(G, \psi)$ that does not depend on ψ . Each operator $A \in UL^m(G, \psi)$ can for any $\varepsilon > 0$ be decomposed into a sum $A = B_\varepsilon + R_\varepsilon$ with $B_\varepsilon \in UL^m(G)$, $K_{B_\varepsilon}(g, h) = 0$ for $\rho(g, h) > \varepsilon$, and $R_\varepsilon \in UL^{-\infty}(G, \psi)$.

If $m \in \mathbf{C}$, then the subclasses $UL_{cl}^m(G, \psi)$ and $UL_{cl}^m(G)$ of classical pseudodifferential operators of order m are singled out in $UL^{Re\,m}(G, \psi)$ and $UL^{Re\,m}(G)$. Namely, $A \in UL_{cl}^m(G, \psi)$ (or $A \in UL_{cl}^m(G)$) for an operator $A \in UL^{Re\,m}(G, \psi)$ (respectively, $A \in UL^{Re\,m}(G)$) means that all the operators A_g on U_g are classical Ψ DO's of order m (see, for example, §3 in [7]) uniformly with respect to $g \in G$, i.e., the constants in the estimates for all the derivatives of the homogeneous functions in the asymptotic expansion of the symbols $a_g(x, \xi)$ and in the estimates for the remainder terms of these asymptotic expansions can be chosen to be independent of g . For example, if A is a differential operator of order m and $A \in UL^m(G, \psi)$, then $A \in UL_{cl}^m(G)$.

DEFINITION 2. An operator $A \in UL^m(G, \psi)$ is said to be *uniformly elliptic* if all the operators A_g on U_g are elliptic (in the sense of Definition 5.3'' in [7]) uniformly with respect to $g \in G$. The class of all uniformly elliptic operators in $UL^m(G, \psi)$ (respectively, $UL^m(G), UL_{cl}^m(G, \psi), UL_{cl}^m(G)$) is denoted by $EUL^m(G, \psi)$ (respectively, $EUL^m(G), EUL_{cl}^m(G, \psi), EUL_{cl}^m(G)$).

An operator $A \in UL_{cl}^m(G, \psi)$ or $A \in UL_{cl}^m(G)$ is uniformly elliptic if and only if its leading symbol $a_m: T^*G \setminus 0 \rightarrow \mathbf{C}$ has the following property:

$$|a_m(\nu)| \geq \varepsilon > 0, \quad \nu \in T^*G, \quad |\nu| = 1,$$

where ε does not depend on ν , and $|\nu|$ is the length of the vector ν with respect to the left-invariant Riemannian metric on G .

THEOREM 2. a) If $A \in EUL^m(G, \psi)$, then there exists an operator $B \in EUL^{-m}(G)$ such that

$$BA = I - R_1, \quad AB = I - R_2, \quad R_j \in UL^{-\infty}(G, \psi), \quad j = 1, 2.$$

If, moreover, $A \in EUL^m(G)$, then $R_j \in UL^{-\infty}(G) = \bigcap_m UL^m(G)$.

b) For any $s \in \mathbf{R}$ there exists an operator $\Lambda_s \in EUL_{cl}^s(G)$ with positive leading symbol.

We introduce the uniform Sobolev spaces $H^s(G)$. Let

$$H^{-\infty}(G) = \left\{ u: u \in \mathcal{D}'(G), u = \sum_{|\alpha| \leq N} X^\alpha f_\alpha, f_\alpha \in L_2(G) \right\},$$

where N and f_α depend on u . Then for any $s \in \mathbf{R}$

$$H^s(G) = \{u: u \in H^{-\infty}(G), \Lambda_s u \in L_2(G)\}.$$

A Hilbert space structure is introduced on $H^s(G)$ in a way similar to that in [1] (see also [7]). If $A \in UL^m(G, \psi)$, then A extends for any $s \in \mathbf{R}$ to a continuous linear operator $A: H^s(G) \rightarrow H^{s-m}(G)$. There is also a theorem on elliptic regularity: if $A \in EUL^m(G, \psi)$, $u \in H^{-\infty}(G)$, and $Au = f \in H^s(G)$, then $u \in H^{s+m}(G)$.

THEOREM 3. If $m > 0$ and $A \in EUL^m(G, \psi)$ is formally selfadjoint (i.e., symmetric on $C_0^\infty(G)$), then it is essentially selfadjoint and its closure is a selfadjoint operator with domain $H^m(G)$.

Finally, we mention that, though the left and right Haar measures on G coincide because of its unimodularity, the left and right Sobolev spaces, classes UL^m , and so on, do not coincide in general. All the left objects coincide (pairwise) with the corresponding right objects if and only if the left and right spaces $H^1(G)$ coincide, and this happens, in turn, if and only if G is a central extension of a compact group.

2. Suppose that the operator $A \in EUL^m(G, \psi)$ is invertible as an operator $A: H^s(G) \rightarrow H^{s-m}(G)$ for some $s \in \mathbf{R}$. Then the same is true for any s , and we then speak simply of the invertibility of A . In this case we define the Green's function $L(g, h)$ to be the kernel of A^{-1} , so that $L \in \mathcal{D}'(G \times G)$. We now discuss the question of how the Green's function decreases as $\rho(g, h) \rightarrow +\infty$.

THEOREM 4. Let $f: [0, +\infty) \rightarrow (0, +\infty)$ be a smooth nondecreasing continuous function satisfying the following conditions:

- $f(0) = 1$ and $f(r+s) \leq f(r)f(s)$ for all $s, r \geq 0$.
- $\lim_{t \rightarrow +\infty} (f(\varepsilon_0 t) / \psi^N(t)) = 0$ for some $\varepsilon_0 > 0$ and $N > 0$.
- $f^{(k)}(t) / f(t) \leq C_k$ for any integer $k \geq 0$.

If an operator $A \in EUL^m(G, \psi)$ is invertible and L is its Green's function, then there exists an $\varepsilon > 0$ such that for any $r > 0$

$$|X_g^\alpha X_h^\beta L(g, h)| \leq C_{\alpha\beta r} f^{-1}(\varepsilon \rho(g, h)) \quad \text{for } \rho(g, h) \geq r > 0,$$

where α and β are arbitrary multi-indices.

In particular, if A is a differential operator, then we can take $f(r) = e^r$, which gives an exponential decrease for the Green's function. If $\psi(r) = 1 + r$, then we can take $f(r) = 1 + r$, which means that the algebra $\bigcup_m UL^m(G, 1 + r)$ is invariant with respect to taking the operator inverse to an elliptic Ψ DO (this result is applicable, for example, in the case of nilpotent groups).

3. An operator $R: C_0^\infty(G) \rightarrow \mathcal{D}'(G)$ is said to be *infinitely smoothing in the scale of* $H^s(G)$ if for any $s, t \in \mathbf{R}$ the operator R extends to a continuous mapping $R: H^s(G) \rightarrow H^t(G)$. The class of all such operators is denoted by $\tilde{U}L^{-\infty}(G)$. Let

$$\tilde{U}L^m(G) = \{A: A = A_1 + R, A_1 \in UL^m(G), R \in \tilde{U}L^{-\infty}(G)\}.$$

It is easy to see that $UL^m(G, \psi) \subset \tilde{U}L^m(G)$. Let

$$\tilde{U}L_{cl}^m(G) = \{A: A = A_1 + R, A_1 \in UL_{cl}^m(G), R \in \tilde{U}L^{-\infty}(G)\},$$

where the leading symbol a_m of the operator A_1 is called the *leading symbol of the operator* A . Assume that $A_1 \in EUL_{cl}^m(G)$ and that a_m does not take values in some angle Λ with vertex at 0 in the complex plane \mathbf{C} , and some ray beginning at 0 does not intersect the spectrum of A . Then the complex powers A^z of A are defined according to the scheme of Seeley [4].

THEOREM 5. a) $A^z \in \tilde{U}L_{cl}^{mz}(G)$ under the assumptions described above.

b) If, moreover, $\psi(r) = 1 + r$ and $A \in EUL^m(G, \psi)$, then $A^z \in UL_{cl}^{mz}(G, \psi)$.

Suppose that the kernel $K_z(g, h)$ of A^z is continuous on $G \times G$ for $\text{Re } z < -n/m$. There are theorems on analytic extension of this kernel that are completely analogous to the theorems of Seeley in [4]. From this, a standard application of Ikehara's Tauberian theorem gives us the asymptotic behavior of the spectral function of an operator A of the form described in the case when the operator is selfadjoint and its leading symbol is nonnegative.

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