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PSEUDODIFFERENTIAL OPERATORS IN $\mathbf{R}^n$

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In the paper we study a certain class of pseudodifferential operators in $\mathbf{R}^n$, which allows us to prove the normal solvability of a broad class of problems in unbounded regions not requiring the conditions of radiation at infinity. Such problems have been considered previously in various papers among which we note [1,3,5,7,9]. In particular, Grušin's paper [3] contains the construction of a regularizer very similar to the one considered here. However, the approach we have proposed allows us to construct a somewhat more exact regularizer (although in a narrower class of operators) and to make important applications to the study of the defect indices of operators in $\mathbf{R}^n$ and to questions important in quantum mechanics, concerning operators with a small parameter h (see [2,4,8]).

1. The pseudodifferential operator $p(x, (h/i)(\partial/\partial x))$ in $\mathbf{R}^n$ with the symbol $p(x, \xi)$, where $x = (x_1, \cdots, x_n)$, $\xi = (\xi_1, \cdots, \xi_n)$, is given by the formula

$$\left(p\left(x, \frac{h}{i}\frac{\partial}{\partial x}\right) u\right)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} p(x, h\xi) e^{i\langle x, \xi\rangle} \hat{u}(\xi)\, d\xi, \tag{1}$$

where h is some positive constant which, as a rule, is taken equal to 1, $\langle x, \xi\rangle = \sum_{i=1}^n x_i \xi_i$, $\hat{u}(\xi)$ is the Fourier transform of the function $u(x)$:

$$\hat{u}(\xi) = \int_{\mathbf{R}^n} e^{-i\langle x, \xi\rangle} u(x)\, dx.$$

Definition 1. Let m, ρ be real numbers and, moreover, let $0 < \rho \leq 1$. By G_ρ^m we denote the set of $p(y) \in C^\infty(\mathbf{R}^{2n})$ such that the following estimate is fulfilled with certain constants C_γ for all multi-indices γ:

$$|\partial^{|\gamma|} p(y) / \partial y^\gamma| \leqslant C_\gamma (1 + |y|)^{m - \rho|\gamma|}, \quad y \in \mathbf{R}^{2n}. \tag{2}$$

The class G_ρ^m is scarcely broader than the class of symbols introduced in [8]; however, it already permits us to examine in a sufficiently meaningful manner the division problem referred to in [8]. We note the relation of the class G_ρ^m with the Hörmander classes [6]:

$$G_\rho^m \subset S_{\rho, 0}^m(\mathbf{R}^n). \tag{3}$$

The following proposition can be verified in the same way as [6].

Proposition 1. *If* $p(x, \xi) \in G_\rho^m$, *then the corresponding operator* $p(x, \partial/\partial x)$ *is a continuous operator from* $S(\mathbf{R}^n)$ *into* $S(\mathbf{R}^n)$ *and can be continued up to a continuous* *mapping of* $S'(\mathbf{R}^n)$ *into* $S'(\mathbf{R}^n)$. *Its generalized kernel in the sense of Schwartz belongs to* $S'(\mathbf{R}^n \times \mathbf{R}^n)$ *and is infinitely differentiable outside the diagonal* $\mathbf{R}^n \times \mathbf{R}^n$.

We remark here only that $p(x, \partial/\partial x)u$ is defined, for $u \in S'$, by the equality

$$(p(x, \partial / \partial x)u, \varphi) = (u, p^*(x, \partial / \partial x)\varphi), \quad \varphi \in S,$$

where $p^*(x, \partial/\partial x)$ is the operator formally adjoint to $p(x, \partial/\partial x)$.

Theorem 1 (The composition formula). *If* $a(x, \xi) \in G^{m_1}_\rho$, $b(x, \xi) \in G^{m_2}_\rho$, *then*

$$a\left(x, \frac{h}{i}\frac{\partial}{\partial x}\right) b\left(x, \frac{h}{i}\frac{\partial}{\partial x}\right) u = c_h\left(x, \frac{h}{i}\frac{\partial}{\partial x}\right) u$$

where $c_h(x, \xi) \in G^{m_1+m_2}_\rho$ *is given by the formula*

$$c_h(x, \xi) = a\left(x, \xi + \frac{h}{i}\frac{\partial}{\partial x}\right) b(x, \xi) \tag{4}$$

and admits of the following expansion for any positive integer N:

$$c_h(x, \xi) = \sum_{|\alpha| \leqslant N-1} \frac{(ih)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|} a(x, \xi)}{\partial \xi^\alpha} \frac{\partial^{|\alpha|} b(x, \xi)}{\partial x^\alpha} + h^N r_N(x, \xi, h), \tag{5}$$

where $r_N(x, \xi, h) \in G^{m-\rho N}_\rho$ *with respect to* (x, ξ) *and, moreover, the constants in estimates of form* (2) *for* r_N *do not depend on* h.

Proof. Let us derive an estimate of the remainder term r_N. From the properties of the terms in formula (5) it follows that it is sufficient to find for any N_0 an N such that $r_N \in G^{m-\rho N_0}_\rho$ with constants in estimates (2), which do not depend on h. By writing the remainder term r_N as the remainder in the integral form of the Taylor formula, we get that it is enough to learn how to estimate uniformly in t, $0 < t \leq 1$, integrals of the form

$$I = \iint e^{i\langle x-z, \zeta\rangle} b(z, \xi) a(x, \xi + t\zeta)\, dz\, d\zeta, \tag{7}$$

where $a \in G^{m'}_\rho$, $b \in G^{m''}_\rho$ and, moreover, m' and m'' are sufficiently small.

Lemma. *Let* $\tilde{b}(\zeta, \xi) = \int e^{-i\langle z, \zeta\rangle} b(z, \xi)\, dz$. *Then the estimate*

$$|\tilde{b}(\zeta, \xi)| \leqslant C_p |\zeta|^{-p} (1 + |\xi|)^{m''+n-\rho p}$$

is satisfied when $m'' + n - \rho p < 0$.

Proof of the lemma. If γ is a multi-index with $|\gamma| = p$, then we get

$$|\zeta^\gamma \tilde{b}(\zeta, \xi)| = \left|\int e^{-i\langle z, \zeta\rangle} \partial_z^\gamma b(z, \xi)\, dz\right| \leqslant \int (1 + |z| + |\xi|)^{m''-\rho p} dz$$
$$= C_p (1 + |\xi|)^{m''+n-\rho p},$$

where $C_p = \int (1 + |\eta|)^{m''-\rho p}\, d\eta$. Whence follows the lemma's assertion.

Let us go on to estimate integral (7). We split it up into two parts,

$$I = I_1 + I_2 = \iint_{|\zeta| \leqslant |y|/2} \ldots + \iint_{|\zeta| \geqslant |y|/2} \ldots,$$

where $y = (x, \xi)$. In integral I_1 we have the estimate

$$|a(x, \xi + t\zeta)| \leqslant C(1 + |y|)^{m'},$$

whence when $m'' < -n$ we obtain

$$|I_1| \leqslant C(1 + |y|)^{m'+n}(1 + |\xi|)^{m''+n} \leqslant C(1 + |y|)^{m'+n}.$$

We make use of the lemma to estimate I_2. When $m' < 0$ we obtain

$$|I_2| \leqslant C \int\limits_{|\zeta| \geqslant |y|/2} |\tilde{b}(\zeta, \xi)|\, d\zeta \leqslant C \int\limits_{|\zeta| \geqslant |y|/2} |\zeta|^{-p}\, d\zeta \cdot (1 + |\xi|)^{m''+n-\rho p} \leqslant C|y|^{-p+n}(1 + |\xi|)^{m''+n-\rho p} \leqslant C_p |y|^{-p+n}$$

when p is sufficiently large.

The estimates obtained yield the desired estimate for I when $|y| \geq 1$. The estimate when $|y| \leq 1$ is obtained from the obvious equality

$$I = \iint \frac{a(x, \xi + t\zeta)}{(1 + |\zeta|^2)^M} [(1 - \Delta_z)^M b(z, \xi)]\, e^{i\langle x-z, \zeta\rangle}\, dz\, d\zeta,$$

where Δ_j is the Laplace operator in z, so that the integral already is convergent when $m'' < -n$ and for sufficiently large M and we have the estimate, uniform in t,

$$|I| \leqslant C \quad \text{for} \quad |y| \leqslant 1.$$

Theorem 1 is proved.

We can analyze analogously the question of the symbol of the adjoint operator and its expansion in powers of h.

2. Let us now consider the question of the inverse operator and of the regularizer.

Definition 2. By Γ^m_ρ we shall denote a subset of G^m_ρ consisting of the symbols $p(y)$ for which the estimate

$$|\partial^\gamma_y p(y)/p(y)| \leqslant C_\gamma (1 + |y|)^{-\rho|\gamma|}, \quad |y| \geqslant M \tag{8}$$

is fulfilled.

Condition (8) is an obvious analog of hypoellipticity in the sense of Hörmander [6] and, analogous to [6], this condition can be written for the matrix symbols $p(y)$. The next theorem is obtained by a literal repetition of the arguments presented in [6].

Theorem 2. *If* $p(y) \in \Gamma^m_\rho$, *then there exists a symbol* $q(y) \in G^m_\rho$ *such that the relations*

$$q(x, \partial/\partial x)p(x, \partial/\partial x)u = u + T_1 u, \quad p(x, \partial/\partial x)q(x, \partial/\partial x)u = u + T_2 u,$$

hold, where the operators T_1 *and* T_2 *have kernels* $K(x, y)$ *belonging to* $S(\mathbf{R}^n \times \mathbf{R}^n)$ *and, hence, map the whole* $S'(\mathbf{R}^n)$ *into* $S(\mathbf{R}^n)$.

Corollary. *If* $p(y) \in \Gamma^m_\rho$, *then the operator* $p(x, \partial/\partial x)\colon S \to S$ *has a kernel and a cokernel of finite dimensions and from the fact that*

$$p(x, \partial/\partial x)u = f,$$

where $f \in S$, $u \in S'$ *it follows that* $u \in S$. *In particular, if* $u \in S'$ *and* $p(x, \partial/\partial x)u = 0$, *then* $u \in S$.

Remark. Hörmander [10] has answered in a similar way the question on the index of the operator $p(x, \partial/\partial x)$ with $p(y) \in \Gamma_\rho^m$.

In Theorem 2 we had taken $h = 1$. Let us now consider what influence the introduction of a small parameter h has.

Theorem 3. *Let $p(y) \in \Gamma_\rho^m$ and $p(y) \neq 0$ for any $y \in \mathbf{R}^n$ whatsoever. Then, for small h the operator $p(x, (h/i)(\partial/\partial x))$ is invertible and the inverse operator has the form $q_h(x, (h/i)(\partial/\partial x))$ with some $q_h(y) \in G_\rho^M$ admitting of an asymptotic expansion in powers of h,*

$$q_h(y) = q_0(y) + hq_1(y) + \ldots + h^{N-1}q_{N-1}(y) + h^N r_N(y; h), \tag{9}$$

where $q_0(y) = (p(y))^{-1}$ and $r_N(y, h) \in G_\rho^{s(N)}$ with constants in estimates (2), not depending on h, and here $s(N) \to -\infty$ as $N \to +\infty$.

This theorem is a simple corollary of Theorems 1 and 2. We remark only that the symbol $q_h(x, \xi)$ is constructed by the natural method of successive approximations, starting with $q_0(y) = (p(y))^{-1}$.

3. Now let $p(y) \in G_\rho^m$ be the symbol of the formally selfadjoint operator $p(x, \partial/\partial x)$ (i.e., of the operator symmetric on $C_0^\infty(\mathbf{R}^n)$). Let us establish a theorem giving the sufficient conditions for the selfadjointness of the closure of operator $p(x, \partial/\partial x)$, i.e., for the absence of defect indices in the operator $p(x, \partial/\partial x)$.

Theorem 4. *Let $p(y) \pm i \in \Gamma_\rho^m$. Then the defect indices of operator $p(x, \partial/\partial x)$ equal 0.*

Proof. We assume that $u \in L^2(\mathbf{R}^n)$ and $p(x, \partial/\partial x)u = \pm iu$ (the operator on the left is to be taken as being adjoint to the corresponding operator on C_0^∞, i.e., taken in generalized functions, as we have done previously for all $u \in S'$). From the corollary to Theorem 2 we get that $u \in S$, but then $u = 0$ since the operator $p(x, \partial/\partial x)$ is symmetric on S, which follows from its continuity on S and from the fact that C_0^∞ is dense in S. Theorem 4 is proved.

In conclusion we remark that in the class of operators described it is easy to obtain: the necessary and sufficient conditions for complete continuity in $L^2(\mathbf{R}^n)$; conditions for the discreteness of the spectrum; continuity in the natural scale of the spaces, connected with fractional powers of the operator of a harmonic oscillator; the unique solvability of the Cauchy problem in the class of power-growth functions for the corresponding parabolic equations with variable growing coefficients.

Finally, the whole theory can be generalized to the broader class of symbols defined by the estimates

$$|\partial_x^\beta \partial_\xi^\alpha p(x, \xi)| \leqslant C_{\alpha\beta}(1 + |\xi|)^{m_1 - \rho_1|\alpha| + \delta_1|\beta|}(1 + |x|)^{m_2 - \rho_2|\beta| + \delta_2|\alpha|},$$

if only $\delta_1 < \rho_1$ and $\delta_2 < \rho_2$. With certain insignificant modifications all the theorems generalize to the case of the symmetric Weyl symbol.

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