

PSEUDODIFFERENCE OPERATORS AND THEIR INVERSION

UDC 517.43

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1. Let X be a discrete metric space with metric ρ , $B(x_0, r)$ its closed ball of radius r centered at the point x_0 , and $\nabla(x_0, r)$ the number of points in the ball. We assume that $V(x_0, r) \leq V(r)$, where the function $V: [0, +\infty) \rightarrow (0, +\infty)$ is continuous nondecreasing and submultiplicative, i.e.

$$(1) \quad V(r+s) \leq V(r)V(s), \quad r, s \geq 0.$$

It follows in particular that $V(r) \leq Ce^{ar}$ for some C and a . The basic example is a group $X = G$ with a finite number of generators g_1, \dots, g_n equipped with the left invariant metric, in which the distance from the point $g \in G$ to the identity is equal to the length of the smallest word expressing g in terms of g_1, \dots, g_n and $g_1^{-1}, \dots, g_n^{-1}$. Then one can take $V(r) = \exp f(r)$, where $f(r)$ is obtained by linear interpolation of values $f(r) = \ln V(e, r)$, $r \in \mathbb{Z}_+$, on each interval $[n, n+1]$, $n \in \mathbb{Z}_+$. The function $V(r)$ thus constructed grows polynomially as r^d on the lattice \mathbb{Z}^d and exponentially as $(2n-1)^r$ in case G is a free group with generators g_1, \dots, g_n . Grigorchuk [1] showed that there exist groups G where $V(r)$ has an intermediate rate of growth between polynomial and exponential functions.

Let $l^p(X)$ denote the usual Banach spaces of complex valued functions on X with norms

$$\|f\|_p = \left(\sum_{x \in X} |f(x)|^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad \|f\|_\infty = \sup_{x \in X} |f(x)|.$$

LEMMA 1. Let the function $f: X \rightarrow \mathbb{C}$ be such that

$$|f(x)| \leq CV(\rho(x, x_0))^{-1-\epsilon},$$

where $x_0 \in X$, and C and ϵ are positive numbers independent of x . Then $f \in l^1(X)$ and $\|f\|_1 \leq MC$, where $M = M(\epsilon)$ is independent of f, C and x_0 .

2. Let us introduce the weight function $\psi: [0, +\infty) \rightarrow (0, +\infty)$ which is nondecreasing and satisfies

$$(2) \quad \exists N, C: \psi(r+s) \leq C\psi^N(r)\psi^N(s), \quad r, s \geq 0,$$

$$(3) \quad \exists N, C: V(r) \leq C\psi^N(r), \quad r \geq 0.$$

Condition (1) implies, for instance, that $\psi(r)$ can be always taken equal to $V(r)$ or e^{ar} , where $a > 0$. In fact conditions (2) and (3) mean that $\psi(r)$ has growth intermediate between $V(r)$ and e^{ar} . Using the weight function ψ , one can introduce the Schwartz space $S(X, \psi)$, which consists of functions $f: X \rightarrow \mathbb{C}$ such that for a fixed point $x_0 \in X$ and any $N > 0$ there exists $C_N > 0$ such that

$$(4) \quad |f(x)| \leq C_N \psi^{-N}(\rho(x, x_0)).$$

It is easy to see that if (4) holds for some point x_0 then it holds for any other choice of x , so $S(X, \psi)$ is independent of x_0 . The best constant C_N in the right-hand side of (4) defines a seminorm on $S(X, \psi)$ and thus turns it into a Fréchet space. From Lemma 1 and (3) it follows that $S(X, \psi) \subset l^1(X)$ (and consequently $S(X, \psi) \subset l^p(X)$ for any $p \in [1, +\infty]$).

DEFINITION. The class $\mathcal{L}(X, \psi)$ of pseudodifference operators consists of the operators K acting in the space of functions on X such that the matrices $K(x, y)$, $x, y \in X$, of these satisfy the estimate

$$(5) \quad |K(x, y)| \leq C_N \psi^{-N}(\rho(x, y)), \quad N = 1, 2, \dots$$

An operator K is written via its matrix in the usual way:

$$(6) \quad (Kf)(x) = \sum_{y \in X} K(x, y)f(y)$$

and a priori it is defined only on compactly supported functions on X . Then it extends by continuity to all functions f for which the series (6) converges absolutely. In particular, the following assertion is true.

PROPOSITION 1: *If $K \in \mathcal{L}(X, \psi)$, then K extends to operators*

$$K: S(X, \psi) \rightarrow S(X, \psi), \quad K: l^p(X) \rightarrow l^p(X), \quad 1 \leq p \leq \infty.$$

This follows from Lemma 1 and Schur's lemma. One can also easily prove

THEOREM 1 (composition). *If $K_1, K_2 \in \mathcal{L}(X, \psi)$, then $K_1 K_2 \in \mathcal{L}(X, \psi)$.*

Examples of pseudodifference operators on the group G (see §1) are convolutions with finite or rapidly decreasing functions (depending on the rate of growth of ψ), for instance difference operators on the lattice. Toeplitz operators obtained from difference or convolution operators by an orthogonal projection onto a subspace of functions supported in a subset $\Lambda \subset G$ are also pseudodifference operators of the above type.

3. Now we shall proceed to the main problem: inversion of operators K of the type of pseudodifference operators. We want to give a sufficient condition that the inverse operator $L = K^{-1}$ (defined and bounded on some $l^p(X)$ for $p \in [1, +\infty]$) has a matrix $L(x, y)$ satisfying (5).

Let a pair of functions ψ, χ map $[0, +\infty)$ into $(0, +\infty)$, and let the following conditions hold:

A. χ is continuous at 0 and submultiplicative.

B.

$$\exists \varepsilon_0 > 0: \lim_{r \rightarrow +\infty} \frac{\chi(\varepsilon_0 r)}{\psi(r)} = 0.$$

THEOREM 2. *Let the functions ψ and χ satisfy conditions A and B, and let K be an operator on X whose matrix admits the estimate*

$$(7) \quad |K(x, y)| \leq C \psi^{-1}(\rho(x, y)) V^{-1-\delta}(\rho(x, y)),$$

where $C, \delta > 0$. If for some $p \in [1, +\infty]$ the operator $K: l^p(X) \rightarrow l^p(X)$ has a bounded inverse $L = K^{-1}$, then the matrix $L(x, y)$ of the inverse operator satisfies

$$(8) \quad \|\chi(\varepsilon \rho(\cdot, y))L(\cdot, y)\|_p \leq C_1,$$

with some $\varepsilon > 0$ and $C_1 > 0$. In particular,

$$(9) \quad |L(x, y)| \leq C_1 \chi^{-1}(\varepsilon \rho(x, y)), \quad x, y \in X.$$

Let us indicate the scheme of the proof. It exploits the weighted spaces

$$l_{\epsilon,y}^p(X) = \{f: \chi(\epsilon\rho(\cdot, y))f(\cdot) \in l^p(X)\}.$$

The operator K in $l_{\epsilon,y}^p(X)$ is similar to the operator $K_{\epsilon,y}$ in $l^p(X)$, and the latter turns out to be close to K in the operator norm (for small $\epsilon > 0$), and consequently invertible. This immediately implies (8), since $L(x, y) = (L\delta_y)(x)$, where δ_y is the δ -function of the point y (i.e. $\delta_y(y) = 1$, and $\delta_y(x) = 0$ for $x \neq y$); and $\delta_y \in l_{\epsilon,y}^p(X)$, so that the norm of δ_y in $l_{\epsilon,y}^p(X)$ is bounded by a constant independent of y .

Let us remark that essentially the same scheme was used in [2] for the same purpose but in a totally different context: for pseudodifferential operators on \mathbf{R}^n and specific weight functions (polynomial and exponential). Different classes of operators in weighted spaces on \mathbf{R}^n were studied also in [3]–[6].

Let us indicate some specific results that follow from Theorem 2 with a suitable choice of X, ψ, χ and K (or some of these objects).

For an arbitrary X we can take $\chi(r) = e^{-r^\alpha}$, where $0 < \alpha \leq 1$, and $\psi(r) = e^{-r^\alpha}$. Then the estimate

$$(10) \quad |K(x, y)| \leq C \exp[-a\rho(x, y)^\alpha] V^{-1-\delta}(\rho(x, y)),$$

implies

$$(11) \quad |L(x, y)| \leq C_1 \exp[-\epsilon\rho(x, y)^\alpha].$$

In particular this is true if K is an invertible difference operator on the group G , in which case one can take $\alpha = 1$. More generally, if K is an invertible convolution operator on the space $l^p(G)$ for some $p \in [1, +\infty]$ with the function φ on G such that

$$(12) \quad |\varphi(x)| \leq C_a \exp[-a\rho(x, e)]$$

for any $a > 0$, then $L = K^{-1}$ is also a convolution operator with the function φ_1 such that

$$(13) \quad |\varphi_1(x)| \leq C' \exp[-\epsilon\rho(x, e)].$$

The example of $G = \mathbf{Z}$ and a compactly supported function φ , where the convolution operator on $l^2(\mathbf{Z})$ is easily inverted by means of the Fourier transform from $l^2(\mathbf{Z})$ onto $L^2(S^1)$, shows that (13) cannot be replaced with an estimate of the type (12) with an arbitrary $a > 0$ (moreover, $\epsilon > 0$ may be arbitrarily small in this example).

Choosing $\chi(r) = V(r)$, we get that

$$(14) \quad |K(x, y)| \leq CV(\rho(x, y))^{-N}, \quad x, y \in X,$$

where $N > 1$, implies that for any $\kappa > 0$ there exist $\epsilon > 0$ and $C_1 > 0$ so that

$$(15) \quad |L(x, y)| \leq C_1 V^{-N+1+\kappa}(\epsilon\rho(x, y)).$$

In particular, for $X = \mathbf{Z}^d$ this yields the following result: if

$$(16) \quad |K(x, y)| \leq C(1 + \rho(x, y))^{-N},$$

where $N > d$, then for any $\delta > 0$ there exists $C_1 > 0$ so that

$$(17) \quad |L(x, y)| \leq C_1(1 + \rho(x, y))^{-N+d+\delta}.$$

In particular, if $K \in \mathcal{L}(\mathbf{Z}^d, 1 + r)$, then

$$K^{-1} \in \mathcal{L}(\mathbf{Z}^d, 1 + r),$$

so that the algebra $\mathcal{L}(\mathbf{Z}^d, 1 + r)$ is closed under inversion. Of course in deriving (15) from (14) and (17) from (16) as well as (11) from (10) and (13) from (12), we assumed that K is invertible in some $L^p(X)$, $p \in [1, +\infty]$.

Let us remark that in the case $X = \mathbf{Z}^d$ estimates of the type (11) or (17) were proved by Blekher [7] by a different method and with additional assumptions (for instance, for positive or close to identity operators). In [7] some probabilistic applications were given that easily extend to the more general situation considered here.

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Received 1/JULY/83

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Translated by D. GURARIE