

# Lefschetz Fixed Point Formula for Manifolds with Cylindrical Ends

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## Introduction

The Lefschetz fixed point formula which appeared first as a topological device for counting fixed points of maps now plays a role that hardly can be overestimated in various areas of mathematics such as algebraic geometry, arithmetics, representation theory etc. M. Atiyah and R. Bott in [1] treated the problem by utilizing partial differential equations. In [3] the result of [1] was extended to manifolds with boundary, which leads to a classification of boundary fixed points that can be attractive or repulsive. Concerning a very general context for the usual Lefschetz formula (e.g. for CW-complexes and even for general Euclidean neighbourhood retract) the reader may consult A. Dold [7]. A very advanced version of a Lefschetz-type formula in the context of constructible sheafs was recently found by M. Kashikwara [10].

A. V. Efremov [8] proved a Lefschetz fixed point formula for  $L^2$ -cohomologies on infinite Galois coverings of compact manifolds using von Neumann traces since  $L^2$ -cohomologies are infinite-dimensional in the usual sense (but not in the von Neumann sense).

In this paper we investigate the case of manifolds with cylindrical ends, which is probably the first in which the reduced  $L^2$ -cohomologies are finite-dimensional (in the usual sense) in spite of non-compactness of the base manifold (see the paper of M. Atiyah, V. K. Patodi and I. M. Singer [2], where  $L^2$ -cohomologies on manifolds with cylindrical ends naturally appeared, and also a more recent paper of S. Rosenberg [12]).

We start with a  $C^\infty$ -map  $f: (M, N) \rightarrow (M, N)$  with non-degenerate fixed points where  $M$  is a compact manifold with boundary  $\partial M = N$ . Denote by  $\hat{M}$  the manifold which is obtained from  $M$  by attaching cylindrical ends to  $N$ . We consider an extension  $\hat{f}: \hat{M} \rightarrow \hat{M}$  of the map  $f$  and require that  $\hat{f}$  has only non-degenerate fixed points and behaves reasonably at infinity (e.g. it is sufficient to require that  $\hat{f}$  is linear with respect to the coordinate along the axis of the cylinders and has no fixed points near infinity). In Sect. 1 we prove that such an extension exists.

In Sect. 2 we define the Lefschetz number  $L(\hat{f})$  on reduced  $L^2$ -cohomologies by using Kodaira's  $L^2$ -decomposition of the Hilbert space of all square-integrable exterior forms and exponential decay of harmonic  $L^2$ -forms on  $\hat{M}$  which was proved by M. Atiyah, V. K. Patodi and I. M. Singer [2]. We prove that this number depends only on  $f$  but not on the "reasonable" extension  $\hat{f}$ .

The question of calculation of  $L(\hat{f})$  naturally arises and is partially solved in Sect. 3. We consider the case that  $N$  is a disjoint union of spheres  $S^{n-1}$  and obtain a formula

(see Theorem 3.2 and notations before it) which contains not only contributions of fixed points but some global invariants (like the degree of the map  $f$  in the case of connected  $M$ ). With simple arguments concerning the simplest case of a closed interval one can show that there exists no formula which gives an expression of  $L(\hat{f})$  in terms of contributions of fixed points only.

### 1. Preliminaries

Let  $M$  be a compact Riemannian manifold with boundary  $\partial M = N$ ,  $\dim M = \dim N + 1 = n$ . We suppose that  $M$  has a collar near the boundary  $N$  i.e.  $N$  has a neighbourhood in  $M$  of the form  $N \times (\varepsilon, 0]$  with a Riemannian metric  $dn^2 \oplus dt^2$  which is the orthogonal direct sum of a Riemannian metric on  $N$  and the standard metric on  $\mathbf{R}$ . Let us glue to  $M$  the cylinder  $N \times [0, +\infty)$  with the metric  $dn^2 \oplus dt^2$  by identifying  $N \times \{0\}$  in the collar and in the cylinder. So we get a (noncompact) Riemannian manifold  $\hat{M}$ . Such a manifold is called a manifold with cylindrical ends.

Let  $f: (M, N) \rightarrow (M, N)$  be a  $C^\infty$ -map of the pair  $(M, N)$  into itself i.e. a  $C^\infty$ -map  $f: M \rightarrow M$  such that  $f(N) \subset N$ . If  $x \in M$  is a fixed point of  $f$ , i.e.  $f(x) = x$ , then we denote by  $df_x$  the differential. We say that a fixed point  $x$  is non-degenerate if  $\det(1 - df_x) \neq 0$ , i.e. 1 is not in the spectrum of  $df_x$ . We shall always suppose that the following non-degeneracy condition is fulfilled:

$$\text{All the fixed points of } f \text{ are non-degenerate.} \tag{ND}$$

In other words the graph  $\{(x, f(x)) \mid x \in M\}$  is transversal to the diagonal  $\{(x, x) \mid x \in M\}$  in  $M \times M$ . Let us recall the classification of non-degenerate boundary fixed points [3]. Let  $x \in N$ ,  $f(x) = x$ . Then  $df_x$  maps  $T_x N$  into itself and so defines a linear map

$$a_x := df_x \text{ mod } T_x N: T_x M / T_x N \rightarrow T_x M / T_x N \tag{1.1}$$

which can be considered as a real number since  $\dim_{\mathbf{R}} T_x M / T_x N = 1$ .

The non-degeneracy implies that  $a_x \neq 1$ . The point  $x$  is called *attractive* if  $0 \leq a_x < 1$  and *repulsive* if  $a_x > 1$ . This classification is essential to the general Atiyah-Bott-Lefschetz fixed point theorem on manifolds with boundary [3].

We shall work with an extension of  $f$  to  $\hat{M}$  which is denoted by  $\hat{f}$  and satisfies a hypothesis which we shall denote (H1) and which consists of the following four conditions:

- 1)  $\hat{f}$  is a proper  $C^\infty$ -map  $\hat{M} \rightarrow \hat{M}$ ,  $\hat{f}|_M = f$ ;
- 2)  $\hat{f}$  is linear with respect to  $t$  if  $t \geq t_0$ , where  $t \in \mathbf{R}$  is the canonical coordinate along the axis of the cylinder and  $t_0$  is sufficiently large;
- 3) there exists a compact  $K \subset \hat{M}$  and  $\varepsilon > 0$  such that  $\text{dist}(\hat{f}(z, z)) > \varepsilon$  for every  $z \in \hat{M} \setminus K$ ;
- 4) all fixed points of  $\hat{f}$  are non-degenerate.

**Proposition 1.1.** *For every  $C^\infty$ -map  $f: (M, N) \rightarrow (M, N)$  satisfying (ND) there exists an extension  $\hat{f}: \hat{M} \rightarrow \hat{M}$  satisfying (H1).*

*Proof.*

A. Let us introduce necessary notations. We shall use only coordinate neighbourhoods  $U$  on  $M$  which are chosen in such a way that  $f(U)$  is contained in some coordinate neighbourhood. In such coordinates the map  $f$  can be represented by a vector function  $(f^1, \dots, f^n)$  defined on an open set in  $\mathbf{R}^n$ . We shall always assume that the local coordinates in a neighbourhood of a boundary point are chosen in the form  $(y, t)$  where  $y$  is a  $(n - 1)$ -tuple of local coordinates in  $N$  and  $t$  is the canonical normal coordinate. Then  $f^n$  represents a globally defined function on  $N \times (-\varepsilon, 0]$  which we shall also denote by  $f^n$  in order to simplify notations. Obviously the following holds true:

$$f^n(y, 0) = 0 \quad \text{and} \quad f(y, t) \leq 0 \quad \text{if} \quad t \in (-\varepsilon, 0].$$

Now suppose that  $x_0 = f(x_0) \in N$ . Then in local coordinates at  $x = x_0$  it holds  $a_x = \partial f^n / \partial t|_{x=x_0}$  where  $a_x$  is defined in (1.1).

B. We shall start by constructing an extension of  $f^n$  to a  $C^\infty$ -function  $\tilde{f}^n: N \times (-\varepsilon, +\infty) \rightarrow (-\varepsilon, +\infty)$  with the following properties:

- (i) if  $(y, 0)$  is a fixed point of  $f$ , then  $\tilde{f}^n(y, t) \neq t$  for all  $t \geq 0$ ;
- (ii) if  $t \geq 1$ , then  $\tilde{f}^n(y, t) = \alpha(y) t$  with a positive  $C^\infty$ -function  $\alpha: N \rightarrow \mathbf{R}$ ;
- (iii) if  $(y_i, 0)$  is a repulsive (attractive) fixed point of  $f$ , then  $\alpha(y_i) > 1$  (respectively  $\alpha(y_i) < 1$ ).

To do this let us choose a real-valued function  $\varphi \in C^\infty(\bar{\mathbf{R}}_+)$  such that  $\varphi^{(k)}(0) = 0$  for all  $k = 0, 1, \dots$ ;  $\varphi(t) > 0$  on  $\mathbf{R}_+$  and  $\varphi(t) = t$  if  $t \geq 1$ . Using the Borel theorem [9] we can construct a real valued  $C^\infty$ -function  $G: N \times (-\varepsilon, +\infty) \rightarrow \mathbf{R}$  which coincides with  $f^n$  on  $N \times (-\varepsilon, 0]$ . Now let  $\psi \in C_0^\infty(-\varepsilon, +\infty)$ ,  $\psi(t) = 1$  on  $[0, \delta]$ ,  $0 \leq \psi \leq 1$ ,  $\psi(t) = 0$  on  $[2\delta, +\infty)$ , where  $\delta > 0$  is sufficiently small.

Then we can take

$$\tilde{f}^n(y, t) = \begin{cases} f^n(y, t) & \text{if } t \in (-\varepsilon, 0]; \\ \psi(t) G(y, t) + \alpha(y) \varphi(t) & \text{if } t \in (0, +\infty); \end{cases} \tag{1.2}$$

where  $\alpha: N \rightarrow \mathbf{R}$  is an appropriately constructed positive  $C^\infty$ -function. Namely if  $x_i = (y_i, 0)$  is a repulsive fixed point of  $f$ , then we have to choose  $\alpha$  sufficiently large in a small neighbourhood  $V_i \subset N$  of  $y_i$  so that  $\tilde{f}^n(y, t) > t$  for all  $y \in V_i, t > 0$ . If  $x_k = (y_k, 0)$  is an attractive fixed point of  $f$  and  $\delta > 0$  is so small that  $\psi(t) G(y_k, t) < t$  for all  $t > 0$ , then we can choose  $\alpha$  sufficiently small on a neighbourhood  $V_k \subset N$  of  $y_k$  so that  $\tilde{f}^n(y, t) < t$  for all  $y \in V_k, t > 0$ .

If  $y_1, \dots, y_N$  are all boundary fixed points, then we can extend  $\alpha$  to a positive function  $\alpha \in C^\infty(N)$  such that  $\alpha(y) = 1$  on  $N \setminus \bigcup_{j=1}^N \tilde{V}_j$  where  $\tilde{V}_1, \dots, \tilde{V}_N$  are disjoint neighbourhoods of the fixed points  $y_1, \dots, y_N$  such that  $\tilde{V}_j \subset \tilde{V}_j$ .

C. The map  $\hat{f}: \hat{M} \rightarrow \hat{M}$  restricted to a cylinder  $N \times (-\varepsilon_0, +\infty)$  with some  $\varepsilon_0 > 0$  has the image in  $N \times (-\varepsilon, +\infty)$ , hence can be written in the form

$$\hat{f}(y, t) = (g(y, t), \tilde{f}^n(y, t)), \tag{1.3}$$

where  $\tilde{f}^n$  is a scalar function on  $N \times (-\varepsilon_0, +\infty)$  and  $g: N \times (-\varepsilon_0, +\infty) \rightarrow N$ . We shall use the  $\tilde{f}^n$  constructed in B (see (1.2)) and consider  $g$  as a smooth homotopy of  $C^\infty$ -maps  $N \rightarrow N$ . We first construct  $g(\cdot, t)$  for  $t \in (-\varepsilon_0, \delta)$  with some  $\delta > 0$  by using the Borel theorem applied to the composition map  $N \times (-\varepsilon_0, 0] \xrightarrow{g} N \rightarrow \mathbf{R}^d$  (the last arrow is an imbedding of  $N$  in  $\mathbf{R}^d$ ) and taking a projection of a tubular neighbourhood of  $N$  in  $\mathbf{R}^d$  to  $N$ . Then we define

$$g(y, t) \equiv g(y, 0) = -f(y, 0) \quad \text{if } t \geq 1.$$

We can obviously take  $\delta = 1/2$ . Then we can define  $g(y, t) = g(y, 1 - t)$ ,  $t \in (1/2, 1]$ . In this way we obtain a continuous homotopy  $g$  which is not  $C^\infty$  at  $t = 1/2$  and  $t = 1$ . We can also assume that  $g(y, t)$  for  $t \in [0, 1/2]$  has a very small deviation from  $g(y, 0)$  so that all the points  $y$  with  $g(y, t) = y$  are very close to the boundary fixed points of  $f$ . Then an  $\hat{f}$  of the form (1.3) has no fixed points on  $N \times (0, +\infty)$ . Applying a smoothing procedure near  $t = 1/2$  and  $t = 1$  we obtain the desired  $g$ , which completes the proof. ■

## 2. The Lefschetz Number

We shall use the notations  $H^k(X)$ ,  $H^k(X, Y)$  for the usual cohomology groups with real coefficients. Here  $X$  is a manifold which may be compact or noncompact and may have a boundary,  $Y \subset X$  is a submanifold. The group  $H^k(X)$  can be defined as the cohomology group of the De Rham complex

$$0 \rightarrow A^0(X) \rightarrow \dots \rightarrow A^k(X) \rightarrow A^{k+1}(X) \rightarrow \dots \rightarrow A^n(X) \rightarrow 0 \tag{2.1}$$

where  $n = \dim X$ ,  $A^k(X)$  is the space of all smooth  $k$ -forms on  $X$  (which are  $C^\infty$  up to the boundary if  $X$  has a boundary). There is an exact cohomology sequence of the pair  $(X, Y)$

$$\dots \rightarrow H^{k-1}(Y) \rightarrow H^k(X, Y) \rightarrow H^k(X) \xrightarrow{i^*} H^k(Y) \rightarrow H^{k+1}(X, Y) \rightarrow \dots, \tag{2.2}$$

where  $i: Y \rightarrow X$  is the natural inclusion map.

Now let  $X$  be a Riemannian manifold without boundary. Denote by  $A_0^k(X)$  the space of all forms  $\omega \in A^k(X)$  which have a compact support. There is a natural scalar product on  $A_0^k(X)$  induced by the Riemannian metric, so  $A_0^k(X)$  is a prehilbert space. We shall denote the corresponding Hilbert space by  $L^2 A^k(X)$ . If  $X$  is complete, then it has the orthogonal Kodaira decomposition [11]

$$L^2 A^k(X) = \mathcal{H}^k(X) \oplus \overline{dA_0^{k-1}(X)} \oplus \overline{d^*A_0^{k+1}(X)}, \tag{2.3}$$

where the bars denote the closures in  $L^2 A^k(X)$ ,  $d^*$  is the formal adjoint operator to  $d$  and  $\mathcal{H}^k(X)$  is the space of all harmonic  $L^2$ -forms on  $X$ , i.e.

$$\begin{aligned} \mathcal{H}^k(X) &= \{\omega \in A^k(X) \cap L^2 A^k(X) : d\omega = 0, d^*\omega = 0\} \\ &= \{\omega \in A^k(X) \cap L^2 A^k(X) \mid \Delta\omega = 0\}. \end{aligned}$$

Here  $\Delta = dd^* + d^*d$  is the Laplacian induced by the Riemannian metric on  $X$ .

It is also well known that if  $d^w$  denotes the weak extension of  $d$  in  $L^2 A^k(X)$  (i.e. the extension with the maximal domain when  $d$  is defined in the sense of distributions), then

$$\text{Ker } d^w = \mathcal{H}^k(X) \oplus \overline{dA_0^{k-1}(X)}.$$

So it is clear from (2.3) that we have a natural isomorphism

$$\mathcal{H}^k(X) \cong \text{Ker } d^w / \overline{dA_0^{k-1}(X)}$$

(see e.g. [4]). The right-hand side here is usually called the reduced  $L^2$ -cohomology-group (here the word “reduced” refers to the closure which is applied to  $dA_0^{k-1}(X)$ ). So we can consider harmonic  $L^2$ -forms instead of the reduced  $L^2$ -cohomologies.

Now we return to the same notations  $M, N, \hat{M}$  as in Sect. 1.

Denote by  $i: N \rightarrow M$  and  $j: M \rightarrow \hat{M}$  the canonical inclusion maps. We need a representation of  $\mathcal{H}^k(\hat{M})$  given by Atiyah, Patodi and Singer:

**Proposition 2.1** ([2]). *The following sequence of linear maps*

$$0 \rightarrow \mathcal{H}^k(\hat{M}) \xrightarrow{j} H^k(M) \xrightarrow{i^*} H^k(N) \tag{2.4}$$

is exact for every  $k = 0, \dots, n$ .

In particular there is a canonical isomorphism

$$\mathcal{H}^k(\hat{M}) \cong \text{Ker } \{i^*: H^k(M) \rightarrow H^k(N)\}. \tag{2.5}$$

In view of (2.2) we can  $\mathcal{H}^k(\hat{M})$  also identify with the image of  $H^k(M, N)$  in  $H^k(M)$ .

Note that the right-hand side in (2.5) does not depend on the choice of the Riemannian metric on  $M$  and is finite dimensional. It follows that  $\mathcal{H}^k(\hat{M})$  is finite dimensional and its dimension does not depend on the Riemannian metric.

Now let  $\hat{f}: \hat{M} \rightarrow \hat{M}$  be a map which satisfies (H1) and also maps the manifolds  $M$  and  $N$  into themselves, i.e.  $\hat{f}$  defines a map of the triple  $\hat{f}: (\hat{M}, M, N) \rightarrow (\hat{M}, M, N)$ .

Clearly  $\hat{f}$  includes linear maps  $\hat{f}^*: H^k(M) \rightarrow H^k(M)$  and  $\hat{f}^*: H^k(N) \rightarrow H^k(N)$ .

Now we shall define some kind of action of  $\hat{f}$  on  $\mathcal{H}^k(\hat{M})$ . It is proved in [2] that every form  $\omega \in \mathcal{H}^k(\hat{M})$  decays exponentially at infinity on cylinders with all derivatives in natural coordinates of the sort  $(y, t)$  described in Sect. 1. It follows from (H1) that  $\hat{f}^*$  maps  $\mathcal{H}^k(\hat{M})$  to  $L^2 A^k(\hat{M})$ . So we can define a linear map  $P_k \hat{f}^*$  induced by the action of  $\hat{f}^*$  on  $H^k(M)$  and  $H^k(N)$ .

**Proposition 2.2.** *The following diagram is commutative:*

$$\begin{array}{ccccc} 0 & \rightarrow & \mathcal{H}^k(\hat{M}) & \xrightarrow{j^*} & H^k(M) & \xrightarrow{i^*} & H^k(N) \\ & & \downarrow P_k \hat{f} & & \downarrow \hat{f}^* & & \downarrow \hat{f}^* \\ 0 & \rightarrow & \mathcal{H}^k(\hat{M}) & \xrightarrow{j^*} & H^k(M) & \xrightarrow{i^*} & H^k(N) \end{array} \tag{2.6}$$

*Proof.* We need the Hodge decomposition of Sobolev spaces introduced by Dodziuk [6] in a slightly different context. All the arguments advanced in [6] can be applied here

without any changes). Denote

$$A^{v,k} = \{ \omega \in L^2 A^k(\hat{M}) : (I + \Delta)^v \omega \in L^2 A^k(\hat{M}) \},$$

where  $v \in \mathbf{Z}^+$ ,  $I$  is the identity operator, the Laplacian  $\Delta$  is applied in the sense of distributions. In particular

$$A^{0,k} = L^2 A^k(M).$$

Let

$$Z^{v,k} = \{ \omega \in A^{v,k} : d\omega = 0 \}, \quad B^{v,k} = dA^{v+1,k-1}, \quad \mathcal{H}^k = \mathcal{H}^k(\hat{M}).$$

Then  $\mathcal{H}^k \subset A^{v,k}$  for every  $v \in \mathbf{Z}^+$  and

$$Z^{v,k} = \mathcal{H}^k \oplus \overline{B^{v,k}}, \tag{2.7}$$

where the closure of  $B^{v,k}$  is taken with respect to the norm

$$\| \omega \|_{v,k} = ((I + \Delta)^v \omega, (I + \Delta)^v \omega)_0^{1/2},$$

$(\cdot, \cdot)_0$  is the scalar product in  $L^2 A^k(\hat{M})$ , i.e.  $\| \cdot \|_{v,k}$  is the natural norm in  $A^{v,k}$ . The decomposition (2.7) is orthogonal in  $L^2 A^k(\hat{M})$  and also in all the Hilbert spaces  $A^{v,k}$ ,  $v \in \mathbf{Z}^+$ .

The right square of the diagram (2.6) is evidently commutative. The commutativity of the left square is equivalent to the following statement:

$$\text{If } \omega \in \mathcal{H}^k, \text{ then } j^*(\hat{f}^* \omega - P_k \hat{f}^* \omega) \in dA^{k-1}(M). \tag{2.8}$$

First note that the form  $\psi = j^*(\hat{f}^* \omega - P_k \hat{f}^* \omega)$  is closed, i.e.  $d\psi = 0$ . Then it follows from the usual duality arguments [5] that (2.8) is equivalent to the set of equalities:

$$\int_{\gamma} \psi = 0 \quad \text{for every } k\text{-cycle } \gamma \text{ in } M. \tag{2.9}$$

Now let us use (2.7) with  $v$  such that  $v > n/2 - k/4$  so that the Sobolev inequality implies the continuity of every restriction of the forms  $\varphi \in A^{v,k}$  to each simplicial  $k$ -chain  $\gamma$  in  $M$  and also the inequality

$$\sup_{\gamma} |\varphi| \leq C \| \varphi \|_{v,k}, \quad \varphi \in A^{v,k}, \tag{2.10}$$

It is clear that the form  $f^* \omega - P_k f^* \omega \in \overline{B^{v,k}}$  for every  $v \in \mathbf{Z}^+$ . Hence it is the limit of exact forms in  $A^{v,k}$  and due to (2.10) it is a uniform limit of exact forms on every finite  $k$ -chain. So (2.10) implies (2.9) hence (2.8) follows. ■

**Definition 2.3.** The *Lefschetz number* of map  $\hat{f}: (\hat{M}, M, N) \rightarrow (\hat{M}, M, N)$  satisfying (H1) in Sect. 1 is defined as

$$L(\hat{f}) = \sum_{j=0}^n (-1)^j \text{tr } P_j \hat{f}^*.$$

Proposition 2.2 immediately implies

**Corollary 2.4.**  $L(\hat{f})$  depends only on the restriction  $f$  of  $\hat{f}$  to  $M$ , i.e.  $L(\hat{f})$  does not depend on the extension of  $f$  from  $M$  to  $\hat{M}$  provided this extension satisfies (H1).

The question of calculation of  $L(\hat{f})$  naturally arises. It seems natural to conjecture that  $L(\hat{f})$  is the sum of the usual contributions of the fixed points or at least a sum of the contributions of the fixed points with coefficients depending on the nature of the fixed point (interior, attractive, repulsive) which is the case if we consider the Lefschetz number on the absolute cohomologies  $H^k(M)$  or on relative cohomologies  $H^k(M, N)$  [3]. But in fact this statement is false, as easily follows from an elementary consideration of the simplest case  $M = [0, 1]$ .

### 3. Examples

In this section we consider an example suggested in [2]: the case when  $M$  is obtained from a compact, closed, orientable, Riemannian manifold  $\tilde{M}$  by removing a finite number of small open balls. We shall always suppose for the sake of simplicity that every connected component of  $M$  has a nonempty boundary, i.e. there is a removed ball in every connected component of  $\tilde{M}$ . It is clear that  $N = \partial M$  is a disjoint union of  $(n - 1)$ -dimensional spheres. (Here  $n = \dim \tilde{M}$ ).

**Lemma 3.1** ([12]). *There are natural isomorphisms*

$$\mathcal{H}^k = \begin{cases} H^k(\tilde{M}) & \text{if } 0 < k < n; \\ 0 & \text{if } k = 0, n. \end{cases}$$

Now we introduce some necessary notations:

- $l$ : number of fixed connected components for the map  $f$  on  $M$ , i.e. the connected components  $M_1, \dots, M_l$  which are mapped by  $f$  into themselves;
- $r$ : number of fixed spheres of  $f$  on  $N$ , i.e. the number of boundary spheres which are mapped by  $f$  into themselves;
- $f_j$ : restriction of  $f$  to  $M_j$ ;
- $\deg f_j$ : degree of  $f_j: M_j \rightarrow M_j$ , i.e. the sum of  $\pm 1$  over the points  $x \in f_j^{-1}(y)$  for a generic interior point  $y \in M_j$  (such that  $df_x: T_x M \rightarrow T_y M$  is an isomorphism for every  $x \in f_j^{-1}(y)$ ; the sign  $\pm$  depends on whether  $df_x$  preserves a chosen orientation of  $M_j$  or reverses it;
- $F^0$ : set of all interior fixed points of  $f$  (fixed points in  $M \setminus N$ );
- $F_-$ : set of all repulsive fixed points of  $f$  in  $N$ ;
- $F_+$ : set of all attractive fixed points of  $f$  in  $N$ ;
- $v_x$ :  $v_x := \text{sgn det}(1 - df)_x$  for a non-degenerate fixed point  $x$  of  $f$  (i.e.  $v_x$  is the standard contribution of the fixed point  $x$ ).

Now we are ready to formulate the main result.

**Theorem 3.2.** *The Lefschetz number  $L(\hat{f})$  is given by the formula*

$$L(\hat{f}) = \sum_{x \in F^0 \cup F_-} v_x + r - \sum_{j=1}^l [1 + (-1)^n \deg f_j].$$

*Proof.* Clearly neither the left-hand side in (3.1) nor the right one depend on non-fixed connected components. So we can suppose that all the connected components of  $M$  are fixed. Furthermore using the obvious additivity of both sides in (3.1) we see that it is sufficient to consider the case of connected  $M$ . In this case (3.1) takes on the form

$$L(\hat{f}) = \sum_{x \in F^0 \cup F_-} v_x + (r - 1) - (-1)^n \deg f. \tag{3.2}$$

Let  $M$  be obtained from  $\bar{M}$  by removing disjoint balls  $B_1, \dots, B_q$ . Without loss of generality we can suppose that  $\bar{M}$  is a (compact) Riemannian manifold and  $B_1, \dots, B_q$  are flat balls with the radii 1 and with disjoint closures. Let us denote the centers of the balls by  $z_1, \dots, z_q$ . We shall use polar coordinates  $(\vartheta_j, r_j)$  on these balls where  $\vartheta_j \in S^{n-1}$ ,  $0 \leq r_j < 1$ .

**Lemma 3.3.** *There exists an extension of  $f: (M, N) \rightarrow (M, N)$  to a continuous map  $\tilde{f}: \bar{M} \rightarrow \bar{M}$  satisfying the following conditions;*

- (C1)  $\tilde{f} \in C^\infty(\bar{M} \setminus \{z_1, \dots, z_q\})$ ,  $\tilde{f}(\{z_1, \dots, z_q\}) \subset \{z_1, \dots, z_q\}$ ;
- (C2)  $\tilde{f}$  has only non-degenerate fixed points on  $\bar{M} \setminus \{z_1, \dots, z_q\}$ ;
- (C3) the set of all fixed points of  $\tilde{f}$  in  $\bar{M} \setminus (M \cup \{z_1, \dots, z_q\})$  is a set  $F'_+$  which is in one-one correspondence with the set  $F_+$  of all attractive fixed points of  $f$  in  $N$ ; moreover the standard contributions  $v_{x'}$  of the fixed points  $x' \in F'_+$  are opposite to the contributions  $v_x$  of the corresponding fixed points  $x \in F_+$ , i.e.  $v_{x'} = -v_x$ ;
- (C4) there exists  $r' \in (0, 1)$  such that if  $B'_j$  is the ball in  $\bar{M}$  with centre  $z_j$  and radius  $r'$  and  $\tilde{f}(z_j) = z_p$  for some  $j, p \in \{1, \dots, q\}$ , then  $\tilde{f}(B'_j) \subset B'_p$ .

**Proof of Lemma 3.3.** We begin with the construction which was done in the proof of Proposition 1.1. It gives an extension of  $f$  to a map  $\hat{f}: \hat{M} \rightarrow \hat{M}$  such that  $\hat{f}$  has no fixed points on  $\hat{M} \setminus M$  and on the cylinders near infinity

$$\hat{f}(y, t) = (f(y, 0), \alpha(y) t), \quad y \in N, \quad t \geq 1,$$

where  $\alpha$  is a positive  $C^\infty$ -function on  $N$  such that  $\alpha(y) < 1$  if  $y \in F_+$ ,  $\alpha(y) > 1$  if  $y \in F_-$ . Now let us take a positive function  $\beta \in C^\infty(N \times (1/2, +\infty))$  such that

- (i)  $\beta(y, t) = \alpha(y)$  if  $t \in (1/2, 1]$ ;
- (ii)  $\beta(y, t) = \beta_\infty > 1$  if  $t \geq 2$ ;
- (iii)  $\frac{\partial \beta(y, t)}{\partial t} = 0$  if  $t \in (1, 2)$ .

Then we can define another extension  $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$  of  $f$  as follows:

$$\tilde{f} = \begin{cases} \hat{f}(y, t) & \text{if } t \leq 1; \\ (f(y, 0), \beta(y, t) t) & \text{if } t > 1. \end{cases}$$

Now for any boundary sphere  $N_j = \partial B_j$  we can identify the cylinders  $N_j \times (0, +\infty)$  with the pierced balls  $B_j \setminus \{z_j\}$  by identifying  $(y, t)$  with  $(y, (1 + t)^{-1})$ . Then extending  $\tilde{f}$  by continuity we obtain a continuous map  $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$  satisfying (C1)–(C4).

End of the proof of Theorem 3.2. We have to prove (3.2) in the case of connected  $M$ . We shall use a general Lefschetz formula for CW-complexes ([7], Ch. VII, Sect. 6) applied it to  $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$ . This allows us to compute  $L(\tilde{f})$  as the sum of contributions of all fixed points of  $\tilde{f}$ . Summing all the usual contributions of the fixed points in  $F^0 \cup F_- \cup F_+ \cup F'_+$  we get  $\sum_{x \in F^0 \cup F_-} v_x$ . The rest of the fixed points is in a subset  $\{z_1, \dots, z_q\}$ . The locality of the contributions ([7]) implies the possibility of calculating the contribution of the fixed point  $z$  for the restriction  $f|_{B_j}: \bar{B}_j \rightarrow \bar{B}_j$ . Now we can use the homotopy invariance and reduce the calculation to the constant map  $\bar{B}' \rightarrow \bar{B}'_j$  with the image  $z_j$ . This implies that this contribution is 1. So we get

$$L(\tilde{f}) = \sum_{x \in F^0 \cup F_-} v_x + r.$$

On the other hand Lemma 3.1 gives

$$L(\tilde{f}) = L(\hat{f}) + 1 + (-1)^n \operatorname{tr} \tilde{f}|_{H^n(\tilde{M})} = L(\hat{f}) + 1 + (-1)^n \operatorname{deg} f.$$

These two formulas clearly imply (3.2). ■

**Remark.** If  $n = 1$  (i.e.  $M$  is a disjoint union of closed intervals), then  $r = 1$  in (3.1). In particular (3.2) in this case takes on the form

$$0 = L(\hat{f}) = \sum_{x \in F^0 \cup F_-} v_x + \operatorname{deg} f = \sum_{x \in F^0 \cup F_-} v_x + (\varkappa - 1)$$

where  $\varkappa$  is the number of the fixed boundary points of the interval  $\tilde{M}$ . Surely this formula can easily be proved by an elementary consideration of the graph of  $f$ .

### References

- [1] ATIYAH, M. F., BOTT, R.: A Lefschetz fixed point formula for elliptic complexes I, II. *Ann. of Math.* (2) **86** (1967), 374–407; **88** (1968), 451–491.
- [2] ATIYAH, M. F., PATODI, V. K., SINGER, I. M.: Spectral asymmetry and Riemannian geometry I. *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 43–69.
- [3] BRENNER, A. V., SHUBIN, M. A.: The Atiyah-Bott-Lefschetz fixed point theorem for manifolds with boundary. *Funktional. Anal. i Prilozhen.* **15**, N° 4, (1981), 67–68 (Russian).
- [4] CHEEGER, J.: On the Hodge theory of Riemannian pseudo-manifolds. *Proc. Symp. Pure Math.* **36** (1980), 91–146.
- [5] DERHAM, G.: *Variétés différentiables*. Paris: Hermann 1955.
- [6] DODZIUK, J.: DeRham-Hodge theory for  $L^2$ -cohomology of infinite coverings. *Topology* **16** (1977), 57–165.
- [7] DOLD, A.: *Lectures on algebraic topology*. Berlin–Heidelberg–New York: Springer-Verlag 1972.
- [8] EFREMOV, A. V.: The Atiyah-Bott-Lefschetz theorem for Hilbert bundles. *Vestnik Moskov. Univ., Ser. I, Math. Mekh.* 1988, N° 4, 90–93.

- [9] HÖRMANDER, L.: The analysis of linear partial differential operators. I. Berlin—Heidelberg—New York: Springer-Verlag 1983.
- [10] KASHIWARA, M.: Character, character cycle, fixed point theorem and group representations Adv. Stud. Pure Math. **14** (1988).
- [11] KODAIRA, K.: Harmonic fields in Riemannian manifolds. Ann. of Math. **50** (1949), 587—665.
- [12] ROSENBERG, S.: Harmonic forms and  $L^2$ -cohomologies on manifolds with cylinders. Indiana Univ. Math. J. **34** (1985), 335—373.

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