Discreteness of spectrum for the Schrödinger operators on manifolds of bounded geometry

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We consider a Schrödinger operator $H = -\Delta + V(x)$ with a semi-bounded below potential V on a Riemannian manifold M of bounded geometry. A necessary and sufficient condition for the spectrum of H to be discrete is given in terms of V. It is formulated by use of the harmonic (Newtonian) capacity in geodesic coordinates on M. This extends the famous result of A.M. Molchanov [59] where the case $M = \mathbb{R}^n$ was considered.

1. Introduction

Let (M, g) be a Riemannian manifold (i.e. M is a C^∞-manifold, $g = (g_{ij})$ is a Riemannian metric on M), $\dim M = n$. For simplicity of formulations assume that M is connected. Let Δ denote the Laplace-Beltrami operator on scalar functions on M i.e.

$$\Delta u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x^j} \right),$$

where $(x^1, \ldots, x^n)$ are local coordinates, (g^{ij}) is the inverse matrix to g_{ij}, $g = \det(g_{ij})$ and we use the usual summation convention.

The main object of our study is the Schrödinger operator

$$H = -\Delta + V(x) \tag{1.1}$$

where the potential $V = V(x)$ is a real-valued measurable function which is locally in L^2 and globally semi-bounded below i.e.

$$V(x) \geq -C, \quad x \in M, \tag{1.2}$$

with a constant $C \in \mathbb{R}$. We will consider H as a self-adjoint operator in the Hilbert space $L^2(M) = L^2(M, d\mu)$, where $d\mu = \sqrt{g} dx^1 \ldots dx^n$ is the Riemannian volume element. The operator H is determined by the closure of the quadratic form which is a priori defined on the space $C_c^\infty(M)$ of all C^∞ functions with compact support on M.

Let $B(x, r)$ denote the open ball in M with the radius r and the center at x, $\bar{B}(x, r)$ the closure of $B(x, r)$. We will also use the same notation for the balls in a tangent space to M.

In this paper we will assume that (M,g) is a manifold of bounded geometry, i.e. the following two conditions are satisfied:

(a) $r_{inj} > 0$ where r_{inj} is the radius of injectivity of M;

(b) $|\nabla^m R| \leq C_m$, where $\nabla^m R$ is the m-th covariant derivative of the curvature tensor.

The condition (a) is equivalent to the fact that for any $x \in M$ the geodesic exponential map $\exp_x : T_xM \to M$ is defined on a ball $B(0,r) \subset T_xM$ for any $r < r_{inj}$ and provides a diffeomorphism of this ball onto the ball $B(x,r) \subset M$. It follows that the manifold (M,g) is complete. There are normal (geodesic) coordinates in each ball $B(x,r)$ which are induced by the exponential map and some orthonormal coordinates in T_xM. The condition (b) is equivalent to the existence of $r_0 > 0$ such that each derivative of the transition function between two sets of the geodesic coordinates centered at two points $x, x' \in M$ and considered in the balls $B(x,r_0)$ and $B(x',r_0)$, is bounded by a constant which depends only on the order of the derivative ([65]). Throughout the whole paper we will fix a sufficiently small $r_0 \in (0, r_{inj})$, satisfying this condition.

It is easy to extend all the arguments of this paper to the case when V is locally in L^1 (instead of L^2). The corresponding operator should be defined then by the closure of corresponding quadratic form. We will discuss this in more detail in Sect.2.

We can also allow the operator H to be considered on $G = M \setminus S$ where $S \subset M$ is a closed subset (possibly of positive measure). In this case we impose the Dirichlet boundary conditions on ∂G. This means that we should start with the closure of the quadratic form which is defined on $C_c^\infty(G)$ and then take the corresponding operator. This will be discussed in more detail in Sect.6.

Let us say that H has a *discrete spectrum* if its spectrum $\sigma(H)$ consists of eigenvalues of finite multiplicity (with the only accumulation point $+\infty$). We will abbreviate this by writing $\sigma = \sigma_d$.

The main result of the paper is the following

Theorem 1.1. *Assume that (M,g) is a Riemannian manifold of bounded geometry, H is the Schrödinger operator (1.1) with the potential satisfying (1.2). There exists $c > 0$, depending only on (M,g), such that $\sigma = \sigma_d$ if and only if the following condition is fulfilled:*

(D) For any sequence $\{x_k |\ k = 1, 2, \dots\} \subset M$ such that $x_k \to \infty$ as $k \to \infty$, for any $r < r_0/2$ and any compact subsets $F_k \subset \bar{B}(x_k, r)$ such that $\operatorname{cap}(F_k) \leq c r^{n-2}$ in case $n \geq 3$ and $\operatorname{cap}(F_k) \leq c \left(\ln \frac{1}{r}\right)^{-1}$ in case $n = 2$,

$$\int_{B(x_k,r)\setminus F_k} V(x)\,d\mu(x) \to \infty \quad \textit{as} \quad k \to \infty\,. \tag{1.3}$$

Here $\operatorname{cap}(F_k)$ *for* $n \geq 3$ *means the harmonic (or Newtonian) capacity of the set* F_k *in the normal (geodesic) coordinates centered at* x_k*, and for* $n = 2$ *it means the same capacity with respect to a ball* $B(0, R)$ *of fixed radius* $R < r_0$*.*

For the necessary definitions concerning capacity see Section 3.

Note that replacing the Riemannian volume element $d\mu$ in (1.3) by the Lebesgue measure dx in the local geodesic coordinates leads to an equivalent condition.

Remark 1.2. Let us clarify the status of the constant c in Theorem 1.1. It is clear that the condition (D) becomes weaker when c decreases. Therefore this condition is necessary for all sufficiently small $c > 0$. But it is seen from the proof of sufficiency (Sect.5) that it is also sufficient for arbitrarily small $c > 0$. Therefore in the Theorem 1.1 we could replace words

There exists $c > 0$*, depending only on* (M, g)*, such that* $\sigma = \sigma_d$ *if and only if the following condition is satisfied*

by the words

Then $\sigma = \sigma_d$ *if and only if for all sufficiently small* $c > 0$ *the following condition is satisfied*

It is well known (and also follows from Theorem 1.1) that the condition

$$V(x) \to +\infty \quad \text{as} \quad x \to \infty \tag{1.4}$$

implies that $\sigma = \sigma_d$. (For the case $M = \mathbb{R}^n$ this result is due to K. Friedrichs [30].) On the other hand it follows from Theorem 1.1 that $\sigma = \sigma_d$ implies that for any fixed $r > 0$

$$\int_{B(x,r)} V(y)d\mu(y) \to +\infty \quad \text{as} \quad x \to \infty. \tag{1.5}$$

This condition is also sufficient (hence necessary and sufficient) for the spectrum of the one-dimensional Schrödinger operator $H = -d^2/dx^2 + V(x)$ in $L^2(\mathbb{R})$ to be discrete. This was proved by A.M. Molchanov [59] (see also [32]).

A.M. Molchanov [59] treated also the case $M = \mathbb{R}^n$ with $n \geq 3$ and in this context he established the result of Theorem 1.1. A.M. Molchanov claimed that the case $n = 2$ can be settled similarly but in fact it seems that the main necessity argument in [59] needs a substantial modification for $n = 2$.

In early 1960th V.G. Maz'ya discovered new powerful technique in embedding theorems for functional spaces (see short notes [48], [49], [50], [51] and also [52], [53], [54], [57] and especially [55], [56] for more details and further developments). This technique is based on use of isoperimetric inequalities applied on level sets of the functions under consideration. In particular, the paper [50] contains an inequality which implies the Cheeger's lower bound [15] for the smallest Dirichlet eigenvalue of the Laplacian in domains on Riemannian manifolds (see more details in Sect.3 of [56]). M.S.Birman established in [5] (see also [8]), that various spectral statements such as positivity, semiboundedness, discreteness or finiteness of the negative spectrum, discreteness of the whole spectrum can be formulated as some embedding theorems. Accordingly the above mentioned results of V.G. Maz'ya imply these spectral results for the operators of higher order on $\mathbb{R}^n$ (see [50], [52], [53] and especially Sections 2.5 and 12.5 in [55]), and in particular provide another proof of the Molchanov theorem for arbitrary $n \geq 2$ and also similar results for higher order operators (see Maz'ya's papers [52], [53] and his book [55]). A particular case of Maz'ya result for the higher order operators when the capacity is not needed (the case $2l > n$, where $2l$ is the order of the operator), was established earlier by M.S. Birman and B.S. Pavlov [6].

Later V.G. Maz'ya and M. Otelbaev [57] (see also Sect.12.5 in [55]) established even stronger result: 2-sided estimates for the bottom of the essential spectrum through so-called "inner diameter" which is defined in terms of an appropriate capacity.

We will provide more references and a brief review of related results in Sect.6.

We give a self-contained proof of Theorem 1.1 which does not rely on results from [59], though we used many Molchanov's arguments, improving them and correcting when this was needed. In particular we extend Molchanov's arguments to the case $n = 2$.

We choose to follow Molchanov's method because it seems to us simpler for a self-contained exposition.

In Section 2 we describe localization technique which reduces the problem to estimates of the bottom of the Dirichlet or Neumann spectrum of H in balls of a fixed small radius.

In Section 3 we discuss definitions and preliminaries on the harmonic (Newtonian) capacity.

In Section 4 we establish that the condition (D) is necessary for the spectrum of H to be discrete.

In Section 5 we prove that this condition is sufficient.

Finally in Section 6 we formulate and prove corollaries, generalizations and applications of the main theorem, and also give a brief review of most closely related results. We prove that replacing capacity by the Lebesgue measure in Theorem 1.1 gives a sufficient condition for the discreteness of the spectrum. We also formulate and sketch the proof of a generalization of Theorem 1.1 to the case when H is considered not on M but on an arbitrary open subset in M with the Dirichlet boundary condition. In particular this gives a necessary and sufficient condition for the Dirichlet spectrum of $-\Delta$ to be discrete on an open subset in M. (In case $M = \mathbb{R}^n$ these results again go back to A.M. Molchanov [59] and V.G. Maz'ya - see e.g. [54], [55].)

The main result of this paper was announced in [40].

2. Localization

In this section we will show that the discreteness of the spectrum for the Schrödinger operator H is equivalent to some estimates of the Dirichlet or Neumann spectrum of H on the balls of a fixed sufficiently small radius, or to the discreteness of the Dirichlet or Neumann spectrum of H on any infinite disjoint union of such balls.

Let (M, g) be a complete Riemannian manifold. We will use the Riemannian norm $|\cdot| = |\cdot|_g$ induced by the metric g both on tangent and cotangent vectors. Namely, for a tangent vector $v = v^j\partial_j \in T_xM$, where $\partial_j = \partial/\partial x_j$, $(x^1, \dots, x^n)$ are local coordinates, we have

$$|v|^2 = g_{ij}v^iv^j,$$

and for a cotangent vector $p = p_jdx^j$

$$|p|^2 = g^{ij}p_ip_j\ .$$

For any $u \in C^\infty(M)$ define its gradient ∇u as the vector field corresponding to du, i.e.

$$(\nabla u)^i = g^{ij}\frac{\partial u}{\partial x_j}\ .$$

Then $|\nabla u| = |du|$.

Let us consider a Schrödinger operator H of the form (1.1) with the potential V such that $V(x) \geq 1, x \in M$. Due to (1.2) we can always assume this without loss of generality.

Let us consider the corresponding quadratic form

$$Q(u,u) = \int_M (|\nabla u(x)|^2 + V(x)|u(x)|^2)d\mu(x). \tag{2.1}$$

The corresponding hermitian form is

$$Q(u,v) = \int_M (\nabla u(x)\cdot\nabla\overline{v(x)} + V(x)u(x)\overline{v(x)})d\mu(x). \tag{2.2}$$

We refer to the books by M.S. Birman and M.Z. Solomyak [7], and M. Reed and B. Simon [64], VIII.6, X.3, for the basics on the theory of quadratic forms.

Let us start with $C_c^\infty(M)$ as the original domain of Q and then take the closure of Q in $L^2(M)$. It is well known that the closure exists. This means that if we take the completion $\mathcal{H}_1$ of $C_c^\infty(M)$ with the norm induced by the scalar product (2.2), then the natural map of $\mathcal{H}_1$ to $L^2(M)$ is injective. To prove this we can for example note first that $\mathcal{H}_1$ is naturally imbedded into the complete Hilbert space $\tilde{\mathcal{H}}_1$ of functions $u \in L^2_{loc}(M)$ such that $\nabla u \in L^2_{loc}(M)$ (here ∇u is understood in the sense of distributions) and

$$\int_M |\nabla u(x)|^2 d\mu(x) < \infty, \int_M V(x)|u(x)|^2 d\mu(x) < \infty,$$

and second, that $\tilde{\mathcal{H}}_1$ is naturally imbedded in $L^2(M)$. In fact we have the following well known

Lemma 2.1. *In the notations above $C_c^\infty(M)$ is dense in $\tilde{\mathcal{H}}_1$, so $\tilde{\mathcal{H}}_1 = \mathcal{H}_1$.*

Sketch of the proof. Using appropriate smooth or Lipschitz cut-off functions we can prove that functions with compact support are dense in $\tilde{\mathcal{H}}_1$. Let u be such a function. We would like to approximate it by functions from $C_c^\infty(M)$. We can assume that u is real-valued. For any $N = 1, 2, \dots$ define $u_N(x) = u(x)$ if $|u(x)| < N$, $u_N(x) = N$ if $u(x) \geq N$, $u_N(x) = -N$ if $u(x) < -N$. Then $u_N \to u$ in $\tilde{\mathcal{H}}_1$. Therefore it is sufficient to prove that bounded functions with compact support from $\tilde{\mathcal{H}}_1$ can be approximated by functions from $C_c^\infty(M)$. This can be done by using the standard mollifying and the dominated convergence theorem. □

Let us recall that the operator H associated with the quadratic form (2.1) can be obtained as follows. Its domain $\operatorname{Dom}(H)$ consists of all $u \in \mathcal{H}_1$ such that there exists $f \in L^2(M)$ so that

$$Q(u, v) = (f, v), \quad \text{for all } v \in \mathcal{H}_1, \tag{2.3}$$

and then $Hu = f$. Here $(\cdot,\cdot)$ means the scalar product in $L^2(M)$. It is clearly sufficient to require (2.3) to be true for all $v \in C_c^\infty(M)$. Then we can rewrite the equation there as $-\Delta u + Vu = f$ where Δ is applied in the sense of distributions. Now it is clear that we can also describe $\operatorname{Dom}(H)$ as the set of all $u \in \mathcal{H}_1$ such that $-\Delta u + Vu \in L^2(M)$.

Note also that $\mathcal{H}_1 = \operatorname{Dom}(H^{1/2})$.

These considerations are not needed if we have $V \in L^2_{loc}(M)$. In this case we can define our operator on $C_c^\infty(M)$ and it will be essentially self-adjoint, which can be proved by the use of an appropriate modification of the Kato inequality (see [37] or X.4 in [64] where the case $M = \mathbb{R}^n$ is considered). In case when $V \in L^\infty_{loc}(M)$ more general results (for manifolds) can be found e.g. in [18], [19], [21], [61], [62], [71], [72].

Denote

$$\begin{aligned} \mathcal{L} &= \{u \in C_c^\infty(M) |\ \int_M (|\nabla u|^2 + V|u|^2) d\mu \leq 1\} \\ &= \{u \in C_c^\infty(M) |\ (Hu, u) \leq 1\}\ . \end{aligned} \tag{2.4}$$

Lemma 2.2. *$\sigma = \sigma_d$ if and only if $\mathcal{L}$ is precompact in $L^2(M)$.*

Proof. Clearly the spectrum is discrete for H if and only if it is true for $H^{1/2}$, which in turn is equivalent to the compactness of the imbedding $\mathcal{H}_1 \subset L^2(M)$. Now note that it follows from Lemma 2.1 that $\mathcal{L}$ is dense in the unit ball of

$\mathcal{H}_1$. Hence the precompactness of this unit ball in $L^2(M)$ is equivalent to the precompactness of $\mathcal{L}$ which proves the Lemma.

Note that in case $V \in L^2_{loc}(M)$ the result follows also from the essential self-adjointness of H. □

Lemma 2.3. *$\sigma = \sigma_d$ if and only if for any $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{M\setminus B(x_0,R)} |u|^2 d\mu < \varepsilon \text{ for any } u \in \mathcal{L}, \tag{2.5}$$

or, in other words,

$$\int_{M\setminus B(x_0,R)} |u|^2 d\mu \to 0 \text{ as } R \to \infty, \text{ uniformly in } u \in \mathcal{L} .$$

Here x_0 is an arbitrarily fixed point in M.

Proof. The set of restrictions of the functions $u \in \mathcal{L}$ to any fixed ball $B(x_0, R)$ is precompact in $L^2(B(x_0,R), d\mu)$ due to the Sobolev compactness of imbedding theorem. It remains to notice that precompactness of a set $\mathcal{L} \subset L^2(M)$ is equivalent to the precompactness of all such restriction sets together with the condition (2.5). □

Let Ω be an open subset in M. Define the following numbers depending on Ω:

$$\lambda(\Omega) = \inf\left\{ \frac{\int_\Omega (|\nabla u|^2 + V|u|^2) d\mu}{\int_\Omega |u|^2 d\mu},\ u \in C_c^\infty(\Omega) \setminus 0 \right\} , \tag{2.6}$$

$$\mu(\Omega) = \inf\left\{ \frac{\int_\Omega (|\nabla u|^2 + V|u|^2) d\mu}{\int_\Omega |u|^2 d\mu},\ u \in C^\infty(\Omega) \setminus 0 \right\} , \tag{2.7}$$

i.e. $\lambda(\Omega)$ and $\mu(\Omega)$ are bottoms of the Dirichlet and Neumann spectra respectively, in the usual variational understanding (see e.g. [22], [36]).

In both (2.6) and (2.7) we can restrict ourselves to real-valued functions u, so we will always consider real-valued functions unless specified otherwise.

Proposition 2.4.

(a) If $\sigma = \sigma_d$ then $\lambda(\Omega) \to \infty$ as $\Omega \to \infty$.
(Here $\Omega \to \infty$ means $dist(x_0, \Omega) \to \infty$.)
(b) If $\mu(M \setminus \bar{B}(x_0, R)) \to \infty$ as $R \to \infty$ then $\sigma = \sigma_d$.

Proof. (a) Assume that $\sigma = \sigma_d$. Choose an arbitrarily small $\varepsilon > 0$ and assume that $\Omega \subset M \setminus \bar{B}(x_0, R)$ where $R = R(\varepsilon)$ corresponds to ε according to (2.5). Then for any $u \in C_c^\infty(\Omega)$ with $(Hu, u) \le 1$ we should have $(u, u) < \varepsilon$. Therefore

$$(u, u) \le \varepsilon (Hu, u), \quad u \in C_c^\infty(\Omega) ,$$

or

$$\frac{(Hu,u)}{(u,u)} \geq \frac{1}{\varepsilon}, \quad u \in C_c^\infty(\Omega) \setminus \{0\} \, .$$

It follows from (2.6) that $\lambda(\Omega) \geq 1/\varepsilon$. This proves that $\lambda(\Omega) \to \infty$ as $\Omega \to \infty$.

(b) Assume that $\mu(M \setminus \bar{B}(x_0, R)) \to \infty$ as $R \to \infty$. According to (2.7) this implies that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{M \setminus \bar{B}(x_0,R)} (|\nabla u|^2 + V|u|^2) d\mu \geq \frac{1}{\varepsilon} \int_{M \setminus \bar{B}(x_0,R)} |u|^2 d\mu$$

for any $u \in \mathrm{Dom}\,(H)$. Recalling that $V \geq 1$, we deduce that

$$\int_{M \setminus \bar{B}(x_0,R)} |u|^2 d\mu \leq \varepsilon, \quad u \in \mathcal{L} \, .$$

This implies $\sigma = \sigma_d$ due to Lemma 2.3. □

Our next goal will be to give conditions of the discreteness of the spectrum in terms of the behavior of Dirichlet and Neumann spectrum on small balls. The following well known geometric lemma will provide us with convenient coverings of M by balls.

Let $M = \cup_{j \in J} U_j$ be a covering of M by open sets U_j. We define the *multiplicity* of this covering as the maximum possible number N of different $j_1, \ldots, j_N \in J$ such that $U_{j_1} \cap \ldots \cap U_{j_N} \neq \emptyset$.

Lemma 2.5. *There exist $r_0 > 0$ and $N > 0$ such that for any $r \in (0, r_0)$ there exists a covering of M by balls $B(x_j, r)$ with the multiplicity of this covering not greater than N.*

PROOF. We will use an idea of Gromov [33] (see also [71] and [58]). Let us choose a maximal subset $S = \{x_1, x_2, \ldots\} \subset M$ such that $d(x_i, x_j) \geq r$ if $i \neq j$. (Here $d(x,y)$ means the Riemannian distance between x and y.) This can be done by the Zorn lemma. Then for every $x \in M$ there exists j such that $d(x, x_j) < r$ because otherwise we could add x to S which contradicts the maximality of S. This means that $M = \cup_j B(x_j, r)$.

Let us show that the multiplicity N of the covering of M by the balls $B(x_j, r)$ is bounded if $r < r_0$ where r_0 is sufficiently small. Assume that

$$x \in B(x_{j_1}, r) \cap \ldots \cap B(x_{j_N}, r).$$

Then $B(x, 2r) \supset B(x_{j_s}, r/2)$ for all $s = 1, \ldots, N$, and on the other hand the balls $B(x_{j_1}, r/2), \ldots, B(x_{j_N}, r/2)$ are disjoint by the triangle inequality. Therefore

$$N \leq \frac{\sup_{x \in M} \mathrm{vol}\,(B(x, 2r))}{\inf_{x \in M} \mathrm{vol}\,(B(x, r/2))}.$$

Here the right hand side is a constant (independent of r) in $\mathbb{R}^n$ with the standard metric. It is bounded in M if $r \in (0, r_0)$ and r_0 is sufficiently small. Indeed, the Jacobi matrices of the exponential maps and their inverses are uniformly bounded on small balls (see [65]), hence the volumes of small balls in M are estimated from both sides by the volumes of the corresponding balls in the tangent spaces (with the constants which are independent of r). □

Remark 2.6. The bounded geometry requirement is redundant for Lemma 2.5 to hold. In fact, an estimate from above for the ratio of the volumes of the balls holds if we only require that the Ricci curvature is bounded below (see e.g. [16] or [70], Theorem 1.3).

Now we will show that the domains Ω and $M \setminus \bar{B}(x_0, R)$ in Proposition 2.4 can be replaced by the balls $B(x, r)$ of arbitrarily fixed small radius r with $x \to \infty$.

Proposition 2.7.

(a) If $\sigma = \sigma_d$, then $\lambda(B(x, r)) \to \infty$ as $x \to \infty$ for any fixed $r \in (0, r_0/2)$.

(b) If $\mu(B(x, r)) \to \infty$ as $x \to \infty$ for some fixed $r \in (0, r_0/2)$, then $\sigma = \sigma_d$.

PROOF. (a) obviously follows from (a) in Proposition 2.4.

To prove (b) fix $r \in (0, r_0/2)$ and choose a covering of M by balls $B(x_k, r)$, $k = 1, 2, \dots$, as in Lemma 2.5, so that the multiplicity of this covering is $\le N$. Denote

$$\mu_R = \inf\{\mu(B(x_k, r)) |\ B(x_k, r) \cap (M \setminus B(x_0, R)) \neq \emptyset\}\ .$$

Clearly the condition in (b) implies that $\mu_R \to \infty$ as $R \to \infty$.

Denote

$$I_R = \{k |\ B(x_k, r) \cap (M \setminus B(x_0, R + 2r)) \neq \emptyset\}\ .$$

For any $R > 0$ and any $u \in C^\infty(M \setminus B(x_0, R))$ we have

$$\int_{M\setminus B(x_0,R)} (|\nabla u|^2 + V|u|^2) d\mu \ge \frac{1}{N} \sum_{k \in I_R} \int_{B(x_k,r)} (|\nabla u|^2 + V|u|^2) d\mu \ge$$

$$\frac{1}{N} \mu_R \sum_{k \in I_R} \int_{B(x_k,r)} |u|^2 d\mu \ge \frac{1}{N} \mu_R \int_{M \setminus B(x_0, R+2r)} |u|^2 d\mu\ .$$

It follows that

$$\int_{M\setminus B(x_0,R+2r)} |u|^2 d\mu \le N \mu_R^{-1} \quad \text{for any} \quad u \in \mathcal{L},$$

where $\mathcal{L}$ is defined by (2.4). Hence $\sigma = \sigma_d$ due to Lemma 2.3. □

To proceed further we need a well known Lemma which estimates the L^2-norm of a function in a ball $B = B(0, r) \subset \mathbb{R}^n$ (with the flat metric) by the L^2 norm of this function in a smaller homothetic ball provided we can control the Dirichlet integral of the function. (See e.g. [59] where parallelepipeds were considered instead of balls.) To formulate the Lemma let us introduce the following temporary short notations:

$$\|\psi\| = \|\psi\|_{L^2(B)} = \left(\int_B |\psi|^2 dx\right)^{1/2}, \quad \|\psi\|_t = \|\psi\|_{L^2(B_t)},$$

where $0 < t \le 1$, dx is the Lebesgue measure on $\mathbb{R}^n$, $B_t = B(0, tr)$. Similarly define

$$\begin{aligned} &\|\nabla\psi\| = \|\nabla\psi\|_{L^2(B)} = \left(\int_B |\nabla\psi|^2 dx\right)^{1/2}, \\ &\|\nabla\psi\|_t = \|\nabla\psi\|_{L^2(B_t)}. \end{aligned}$$

Lemma 2.8. *The following estimates hold true:*

$$\|\psi\| \le t^{-n/2}\|\psi\|_t + 2^{n+1}r(1-t)\|\nabla\psi\|, \quad t \in [1/2, 1]; \tag{2.8}$$

$$\|\psi\|^2 \le 2t^{-n}\|\psi\|_t^2 + 2^{2n+3}r^2(1-t)^2\|\nabla\psi\|^2, \quad t \in [1/2, 1]. \tag{2.9}$$

In these estimates we assume that ψ is a real-valued function from $C^\infty(B)$.

Proof. To prove (2.8) denote $\psi_t(x) = \psi(tx)$ and consider the function

$$f(t) = \int_B \psi^2(tx)dx = t^{-n}\int_{B_t}\psi^2(x)dx = t^{-n}\|\psi\|_t^2,$$

where $0 < t \le 1$. Differentiating it we get for $t \in (0, 1)$

$$f'(t) = \int_B 2\psi_t(x)(x\cdot\nabla\psi(tx))dx = 2t^{-1}\int_B \psi_t(x\cdot\nabla\psi)_t dx.$$

Using the Cauchy-Schwarz inequality we obtain

$$|f'(t)|^2 \le 4t^{-2}\|\psi_t\|^2\|(x\cdot\nabla\psi)_t\|^2 = 4t^{-2n-2}\|\psi\|_t^2\|(x\cdot\nabla\psi)\|_t^2 \le 4t^{-2n-2}r^2\|\psi\|^2\|\nabla\psi\|^2.$$

Therefore,

$$|f(1) - f(t)| \le \int_t^1 |f'(\tau)|d\tau \le 2r\|\psi\|\|\nabla\psi\|\int_t^1 \tau^{-n-1}d\tau$$

Since

$$\int_t^1 \tau^{-n-1}d\tau = \frac{1-t^n}{nt^n} \le 2^n(1-t), \ t \in [1/2, 1],$$

we further obtain

$$|f(1) - f(t)| \le 2^{n+1} r(1-t)\|\psi\|\|\nabla\psi\|.$$

In particular,

$$f(1) \le f(t) + 2^{n+1} r(1-t)\|\psi\|\|\nabla\psi\|,$$

or

$$\|\psi\|^2 \le t^{-n}\|\psi\|_t^2 + 2^{n+1} r(1-t)\|\psi\|\|\nabla\psi\|.$$

Rewrite this inequality in the form

$$y^2 - 2ay - b^2 \le 0,\ y = \|\psi\|,\ a = 2^n r(1-t)\|\nabla\psi\|,\ b = t^{-n/2}\|\psi\|_t.$$

It follows that $(y-a)^2 \le a^2 + b^2$, hence $y - a \le \sqrt{a^2+b^2} \le a + b$ and $y \le 2a + b$ which is exactly (2.8)

Clearly (2.9) follows from (2.8) if we use the inequality $(a+b)^2 \le 2(a^2+b^2)$, $a, b \in \mathbb{R}$. □

Now we would like to compare $\lambda(B(x,r))$ and $\mu(B(x,r))$. This is done in the following

Lemma 2.9. *There exist $C_1, C_2 > 0$ depending only on (M,g) such that*

$$\mu(B(x,r)) \le \lambda(B(x,r)) \le C_1\mu(B(x,r)) + C_2 r^{-2}, \tag{2.10}$$

for any $x \in M$, $r \in (0, r_0/2)$.

Proof. A. Note first that the inequality $\mu(\Omega) \le \lambda(\Omega)$ holds for any open set $\Omega \subset M$ due to the definitions of $\lambda(\Omega)$ and $\mu(\Omega)$ (see (2.6), (2.7)). Hence we need to prove only the second inequality in (2.10).

Note that due to the bounded geometry conditions replacing the gradient ∇ and the measure $d\mu$ by the Euclidean gradient and measure in normal coordinates in (2.6) and (2.7) leads to comparable ratios which differ from the ones in (2.6), (2.7) by a bounded factor which is also separated from 0 uniformly in u and $r \in (0, r_0/2)$. Therefore it is sufficient to prove (2.10) in the Euclidean case and for $x = 0$. This was done by A.M. Molchanov [59] (for parallelepipeds instead of balls which does not make any difference) but for the sake of completeness we will reproduce his arguments in our notations.

B. Now we will argue in the ball $B = B(0,r) \subset \mathbb{R}^n$ with the flat metric. Let us choose an arbitrary $\varepsilon > 0$ and according to (2.7) take a real-valued $\psi \in C^\infty(B)$ so that

$$\|\psi\|^2 = 1 \quad \text{and} \quad \int_B (|\nabla\psi|^2 + V|\psi|^2)dx \le \mu(B) + \varepsilon. \tag{2.11}$$

It follows in particular that

$$\int_B |\nabla\psi|^2 dx \le \mu(B) + \varepsilon. \tag{2.12}$$

Now let us make an appropriate choice of t, so as to be able to evaluate $\|\psi\|_t$ from below using (2.9). To this end we would like the estimate

$$2^{2n+3}r^2(1-t)^2\|\nabla\psi\|^2 \le \frac{1}{2}\|\psi\|^2 \tag{2.13}$$

to hold true. According to (2.12) it is sufficient to require that

$$2^{2n+4}r^2(1-t)^2(\mu(B)+\varepsilon) \le 1.$$

The minimal $t \ge 1/2$ satisfying this condition is

$$t = \max\left\{\frac{1}{2}, 1 - \frac{1}{2^{n+2}r\sqrt{\mu(B)+\varepsilon}}\right\}. \tag{2.14}$$

Let us fix this value of t in the subsequent arguments. Then (2.9) and (2.13) imply

$$\int_{B_t} |\psi|^2 dx = \|\psi\|_t^2 \ge \frac{1}{4}t^n \ge \frac{1}{2^{n+2}}. \tag{2.15}$$

C. Let us choose a cut-off function $\chi \in C_c^\infty(B)$ such that $\chi = 1$ on B_t, $0 \le \chi(x) \le 1$ and

$$|\nabla\chi(x)| \le \frac{2}{r(1-t)} \tag{2.16}$$

for all $x \in B$. Then take $\phi = \chi\psi$ to serve as a test function in evaluating $\lambda(B)$ from (2.6). We have then

$$\begin{aligned} &|\nabla\phi|^2 + V\phi^2 = |\chi\nabla\psi + \psi\nabla\chi|^2 + V\chi^2\psi^2 \\ &\le |\nabla\psi|^2 + 2|\chi\psi\nabla\chi\cdot\nabla\psi| + \psi^2|\nabla\chi|^2 + V\psi^2 \\ &\le 2(|\nabla\psi|^2 + V\psi^2 + \psi^2|\nabla\chi|^2). \end{aligned}$$

Integrating this over B and using (2.11) and (2.16) we obtain

$$\int_B (|\nabla\phi|^2 + V\phi^2)dx \le 2\left[\mu(B(0,r)) + \varepsilon + \frac{4}{r^2(1-t)^2}\right].$$

Due to (2.15) we also have

$$\int_B \phi^2 dx \ge \frac{1}{2^{n+2}}.$$

Therefore (2.6) implies

$$\lambda(B) \le 2^{n+3}\left[\mu(B) + \varepsilon + \frac{4}{r^2(1-t)^2}\right] \tag{2.17}$$

Note that (2.14) implies

$$\frac{1}{(1-t)^2} = \max\{4, 2^{2n+4}r^2(\mu(B)+\varepsilon)\} \le 4 + 2^{2n+4}r^2(\mu(B)+\varepsilon).$$

Therefore (2.17) gives

$$\lambda(B) \le 2^{n+3}(1+2^{2n+6})(\mu(B)+\varepsilon) + 2^{n+7}r^{-2}.$$

Since $\varepsilon > 0$ is arbitrary, this estimate also holds for $\varepsilon = 0$ i.e.

$$\lambda(B) \le 2^{n+3}(1+2^{2n+6})\mu(B) + 2^{n+7}r^{-2}.$$

This proves the desired estimate (2.10). □

Now we are ready to formulate the localization theorem which is the main result of this section.

Theorem 2.10. *Let (M, g) be a manifold of bounded geometry, and H is a Schrödinger operator (1.1) on M with the potential which is semi-bounded below. Then the following conditions are equivalent:*

(a) $\sigma = \sigma_d$ i.e. H has a discrete spectrum;
(b) $\lambda(B(x,r)) \to \infty$ as $x \to \infty$ for any fixed $r \in (0, r_0/2)$;
(c) $\mu(B(x,r)) \to \infty$ as $x \to \infty$ for any fixed $r \in (0, r_0/2)$;
(d) there exists $r \in (0, r_0/2)$ such that $\lambda(B(x,r)) \to \infty$ as $x \to \infty$;
(e) there exists $r \in (0, r_0/2)$ such that $\mu(B(x,r)) \to \infty$ as $x \to \infty$.

PROOF. 1) Obviously (b) implies (d), and (c) implies (e).

2) Due to Lemma 2.9, (b) is equivalent to (c), and (d) is equivalent to (e).

3) Due to Proposition 2.7, (a) implies (b), and (e) implies (a).

4) Considering the graph of all the implications listed in 1)–3), it is easy to conclude that all the conditions (a)–(e) are equivalent. □

3. Capacity

We will use the harmonic (or Newtonian) capacity for compact subsets $F \subset \mathbb{R}^n$ with respect to an open set $\Omega \subset \mathbb{R}^n$ which includes F, so $F \subset \Omega \subset \mathbb{R}^n$. We will always assume that Ω is bounded, connected and has a C^∞ boundary. In fact we only need $\Omega = B(0,R)$, where $B(0,R)$ is the open ball of a fixed radius $R > 0$ centered at 0.

Let us recall the necessary definitions. We will use the variational version as the main definition of the capacity.

Definition 3.1. For any compact set $F \subset \Omega$ define the *harmonic (or Newtonian) capacity* of F with respect to Ω as

$$\text{(3.1)}\ \operatorname{cap}_\Omega(F) = \inf\left\{\int_\Omega |\nabla u|^2 dx|\ u \in C_c^\infty(\Omega), u = 1 \text{ near } F,\ 0 \le u \le 1 \text{ in } \Omega\right\}.$$

Clearly $F_1 \subset F_2$ implies $\operatorname{cap}_\Omega F_1 \le \operatorname{cap}_\Omega F_2$ for any compact subsets $F_1, F_2 \subset \Omega$.

The following lemma allows to approximate the capacity of an arbitrary compact set $F \subset \Omega$ by the capacity of compacts of the form $\bar{U} \subset \Omega$ where U is an open set with a C^∞ boundary, $\bar{U}$ denotes its closure in $\mathbb{R}^n$.

Lemma 3.2. *In the notations of Definition 3.1*

$$\operatorname{cap}_\Omega(F) = \inf\{\operatorname{cap}_\Omega(\bar{U})|\ F \subset U \subset \bar{U} \subset \Omega,\ U \text{ has a } C^\infty \text{ boundary.}\}$$

PROOF. The proof follows immediately from the Definition 3.1. □

Let us list some equivalent versions of the Definition 3.1. Denote by $\operatorname{Lip}(\Omega)$ the set of Lipschitz functions in Ω, i.e. functions $f : \Omega \to \mathbb{R}$ such that

$$|f(x) - f(y)| \le C|x - y|,\ x, y \in \Omega, \tag{3.2}$$

where $C \ge 0$ does not depend on x, y. Note that any function $f \in \operatorname{Lip}(\Omega)$ extends by continuity to $\bar{\Omega}$ (the closure of Ω in $\mathbb{R}^n$), and satisfies (3.2) on $\bar{\Omega}$ with the same constant C, i.e. $\operatorname{Lip}(\Omega) = \operatorname{Lip}(\bar{\Omega})$. It is well known that $f \in \operatorname{Lip}(\Omega)$ implies that all distributional derivatives $\partial f/\partial x_j$ are in $L^\infty(\Omega)$. Vice versa, if all first distributional derivatives of a distribution f are in $L^\infty(\Omega)$ then f as a distribution coincides with a (uniquely defined) Lipschitz function. (See e.g. [55], [73]). Denote by $\operatorname{Lip}_c(\Omega)$ the set of functions $f \in \operatorname{Lip}(\Omega)$ which have a compact support in Ω.

Proposition 3.3. *The conditions*

$$u \in C_c^\infty(\Omega),\ u = 1 \text{ near } F,\ 0 \le u \le 1 \text{ in } \Omega; \tag{3.3}$$

in (3.1) can be replaced (without changing the left hand side) by any of the following sets of conditions:

$$u \in \operatorname{Lip}_c(\Omega),\ u = 1 \text{ near } F,\ 0 \le u \le 1 \text{ in } \Omega; \tag{3.4}$$

$$u \in \operatorname{Lip}(\Omega),\ u = 1 \text{ near } F,\ u|_{\partial\Omega} = 0,\ 0 \le u \le 1 \text{ in } \Omega; \tag{3.5}$$

$$u \in \operatorname{Lip}(\Omega),\ u = 1 \text{ near } F,\ u|_{\partial\Omega} = 0; \tag{3.6}$$

$$u \in \operatorname{Lip}(\Omega),\ u = 1 \text{ on } F,\ u|_{\partial\Omega} = 0,\ 0 \le u \le 1 \text{ in } \Omega; \tag{3.7}$$

$$u \in \operatorname{Lip}(\Omega),\ u \geq 1 \text{ near } F,\ u|_{\partial\Omega} = 0; \tag{3.8}$$

$$u \in \operatorname{Lip}(\Omega),\ u = 1 \text{ on } F,\ u|_{\partial\Omega} = 0; \tag{3.9}$$

$$u \in \operatorname{Lip}(\Omega),\ u = 1 \text{ near } F,\ u|_{\partial\Omega} = 0,\ \Delta u = 0 \text{ in } \Omega \setminus u^{-1}(\{1\}), \tag{3.10}$$

where $u^{-1}(\{1\}) = \{x|\ x \in \Omega,\ u(x) = 1\}$.

PROOF. Equivalence of (3.3) and (3.4) can be easily obtained by the standard mollifying procedure which leads to an approximation of any function $u \in \operatorname{Lip}_c(\Omega)$ by functions $u_k \in C_c^\infty(\Omega)$ so that for the approximating functions we still have $0 \leq u_k \leq 1$ and $u_k = 1$ near F, and also

$$\int_\Omega |\nabla u_k|^2 dx \to \int_\Omega |\nabla u|^2 dx$$

as $k \to \infty$.

Similarly using an appropriate sequence of cut-off functions we come to the conclusion that (3.5) is equivalent to (3.4).

To prove the equivalence of (3.5) and (3.6) we should replace a function u satisfying (3.6) first by $\max(u, 0)$ and then by $\min(u, 1)$ using the fact that the Dirichlet integral decreases under these operations.

The equivalence of (3.6) and (3.7) can be obtained if we approximate u from (3.7) by functions u_k satisfying (3.5). The functions u_k can be obtained e.g. by a procedure suggested by V. G. Mazya in [55], Sect.2.2.1, i.e. as $u_k(x) = \lambda_k(u(x))$, where

$$0 \leq \lambda_k'(t) \leq 1 + \frac{1}{k},\ \lambda(t) = 0 \text{ near } (-\infty, 0],\text{ and } 1 \text{ near } [1, \infty),\ 0 \leq \lambda \leq 1\ .$$

The same approximation leads to the equivalence of (3.8) and (3.9) to the previous conditions.

Finally the equivalence of (3.10) and (3.6) can be obtained if we replace a function u satisfying (3.6) by a Lipschitz function v which satisfies the same conditions as u and besides is harmonic in $\Omega \setminus \bar{U}$ where U is a small neighborhood of F, $U \subset u^{-1}(\{1\})$, such that U has a C^∞ boundary. (This can be done by solving the appropriate Dirichlet problem.) By the well known variational property of the Dirichlet problem we will have then

$$\int_\Omega |\nabla v|^2 dx \leq \int_\Omega |\nabla u|^2 dx,$$

which proves the equivalence of (3.10) and (3.6). □

Corollary 3.4. *If $F = \bar{U} \subset \Omega$ where U has a C^∞ boundary, then*

$$\operatorname{cap}_\Omega(F) = \int_{\Omega\setminus F} |\nabla u|^2 dx = \int_{\partial U} \frac{\partial u}{\partial \nu} d\sigma, \tag{3.11}$$

where u is the solution of the following Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega \setminus F,\ \ u|_{\partial U} = 1, u|_{\partial\Omega} = 0, \tag{3.12}$$

ν is the exterior unit normal vector to $\partial(\Omega \setminus F)$, $d\sigma$ is the (euclidean) area element on the boundary, the derivative $\partial u/\partial\nu$ is understood as $-\partial u/\partial\nu_{in}$ where $\nu_{in} = -\nu$ is the interior unit normal vector.

Proof. The second equality in (3.11) follows from the divergence formula

$$\int_{\Omega\setminus F} |\nabla u|^2 dx = \int_{\Omega\setminus F} [\operatorname{div}(u\nabla u) - u\Delta u] dx = \int_{\Omega\setminus F} \operatorname{div}(u\nabla u) dx = \int_{\partial(\Omega\setminus F)} u\frac{\partial u}{\partial\nu} d\sigma$$

and the boundary conditions in (3.12). The first equality then follows by an easy limit transition over a neighborhoods of F with C^∞ boundary as in (3.10). $\square$

This corollary allows in particular to calculate the capacity of a ball F with respect to a bigger ball with the same center. Namely, for $U = B(0,r)$ and $\Omega = B(0,R)$ with $r < R$ the solution of the Dirichlet problem (3.12) has the form

$$u(x) = \tfrac{|x|^{2-n} - R^{2-n}}{r^{2-n} - R^{2-n}}, \quad n \geq 3,$$
$$u(x) = \left(\ln \tfrac{R}{r}\right)^{-1} \ln \tfrac{R}{|x|}, \quad n = 2.$$

Using these formulas to calculate the right hand side of (3.11) we easily obtain

$$\operatorname{cap}_{B(0,R)}(\bar{B}(0,r)) = (n-2)\omega_n r^{n-2}(1 - r^{n-2}/R^{n-2})^{-1}, \quad n \geq 3, \tag{3.13}$$

$$\operatorname{cap}_{B(0,R)}(\bar{B}(0,r)) = 2\pi \left(\ln \tfrac{R}{r}\right)^{-1}, \qquad n = 2, \tag{3.14}$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Note that when R is fixed and $r \to 0$, the capacity $\operatorname{cap}_{B(0,R)} \bar{B}(0,r)$ is equivalent to $(n-2)\omega_n r^{n-2}$ for $n \geq 3$ and to $2\pi \left(\ln \frac{1}{r}\right)^{-1}$ for $n = 2$. In case $n \geq 3$ we also have

$$\lim_{R\to\infty} \operatorname{cap}_{B(0,R)}(\bar{B}(0,r)) = (n-2)\omega_n r^{n-2}.$$

Generally for $n \geq 3$ and any compact $F \subset \mathbb{R}^n$ using the maximum principle it is easy to prove the existence of the limit

$$\operatorname{cap}(F) = \lim_{R\to\infty} \operatorname{cap}_{B(0,R)}(F),$$

which is usually called the (absolute) harmonic (or Newtonian) capacity. This absolute capacity can be used instead of the relative one because asymptotically for small r it gives the same result. In case $n = 2$ the $R \to \infty$ limit vanishes, so we really have to use the relative capacity with a fixed $R > 0$.

Remark 3.5. The absolute capacity of a compact $F \subset \mathbb{R}^n$ for $n \geq 3$ can be also defined as follows:

$$\operatorname{cap}(F) = \sup\{m(F) | \frac{1}{(n-2)\omega_n} \int_F \frac{dm(y)}{|x-y|^{n-2}} \leq 1,\ x \in \mathbb{R}^n \setminus F\}\ , \tag{3.15}$$

where m is a finite positive Radon measure on F (a "charge").

This definition corresponds to the physical notion of capacity (in electrostatics), which is the maximal charge which you can "load" on the capacitor so that the potential of this charge is ≤ 1 everywhere.

In case $F = \bar{U}$ where U is a bounded open set in $\mathbb{R}^n$ with a C^∞ boundary, the optimal measure m in (3.15) can be determined as follows. Let us take $u \in \operatorname{Lip}(\mathbb{R}^n)$, such that $u = 1$ on F and u is the solution of the Dirichlet problem similar to (3.12):

$$\Delta u = 0 \text{ in } \mathbb{R}^n \setminus F,\ u|_{\partial U} = 1,\ u(x) \to 0 \text{ as } x \to \infty.$$

Then $m = -\Delta u$ where Δ is understood in the sense of distributions. This means that in fact m is supported on ∂U and has there a density $(\partial u/\partial \nu) d\sigma$.

The corresponding variational definition of the absolute capacity is

$$\operatorname{cap}(F) = \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^2 dx |\ u \in C_c^\infty(\mathbb{R}^n), u \geq 1 \text{ on } F\right\},$$

where u is assumed to be real-valued.

The capacity of a set as a mathematical notion was first introduced by N. Wiener [77] who used it in [78] to establish his famous criterion for regularity of a boundary point with respect to the Dirichlet problem. The most important early developments are due to C.J. de La Vallée-Poussin [75] and O. Frostman [31].

For more details about the capacity we refer to the monographs by D. Adams and L. Hedberg [1], L. Carleson [13], E.M. Landis [41], N.S. Landkof [42], V.G. Maz'ya [55] and J. Wermer [76], as well as to the papers by M.V. Keldysh [38] and V.G. Maz'ya [56].

4. Necessary condition

In this section we will establish that the condition (D) in Theorem 1.1 is necessary for the spectrum of H to be discrete. Let us fix $R > 0$ and denote $\Omega = B(0, R) \subset \mathbb{R}^n$. For simplicity of notations in this section we will write

$\operatorname{cap}(F)$ instead of $\operatorname{cap}_\Omega(F)$ for any compact set $F \subset \Omega$. We will consider the Schrödinger operator (1.1) in the ball Ω (with the usual Euclidean Laplacian).

Our main result in this section will be the following

Theorem 4.1. *If $F \subset \bar{B} \subset \mathbb{R}^n$, where $B = B(0,r)$, $r \leq R/2$, and F is compact, then*

$$\mu(B) \leq 2^{n+4} r^{-2} [\operatorname{cap}(\bar{B})]^{-1} \left(\int_{B \setminus F} V(x)dx + \operatorname{cap}(F) \right), \tag{4.1}$$

provided

$$\operatorname{cap}(F) < 2^{-2n-6} \operatorname{cap}(\bar{B}). \tag{4.2}$$

Here $\operatorname{cap}(\bar{B})$ can be found by the formulas (3.13), (3.14).

Proof. A. Let us choose an arbitrary $\varepsilon > 0$ and take a function $u \in C_c^\infty(\Omega)$ such that $u = 1$ in a neighborhood of F, $0 \leq u(x) \leq 1$ for all $x \in \Omega$, and

$$\int_\Omega |\nabla u|^2 dx \leq \operatorname{cap}(F) + \varepsilon$$

Let us use the function $\psi = 1 - u$ as a test function instead of u in (2.7), so

$$\mu(B) \leq \frac{\int_B (|\nabla \psi|^2 + V\psi^2)dx}{\int_B \psi^2 dx}. \tag{4.3}$$

Note that

$$\int_B |\nabla \psi|^2 dx = \int_B |\nabla u|^2 dx = \int_{B \setminus F} |\nabla u|^2 dx \leq \int_{\Omega \setminus F} |\nabla u|^2 dx \leq \operatorname{cap}(F) + \varepsilon.$$

Since $\psi = 0$ in a neighborhood of F and $0 \leq \psi \leq 1$ everywhere on Ω, we have then

$$\int_B (|\nabla \psi|^2 + V\psi^2)dx \leq \operatorname{cap}(F) + \varepsilon + \int_{B \setminus F} V dx. \tag{4.4}$$

This gives an estimate of the numerator in the right hand side of (4.3). So it remains to estimate the denominator from below.

B. We will start by an estimate from below for $\|\psi\|^2_{L^2(B_2)}$, where $B_2 = B(0, 2r)$. Take the following cut-off function $\theta \in \operatorname{Lip}(\Omega)$:

$$\theta(x) = 0 \text{ on } B, \quad \theta(x) = \frac{|x|}{r} - 1 \text{ on } B_2 \setminus B, \quad \theta(x) = 1 \text{ on } \Omega \setminus B_2.$$

It follows that $|\nabla \theta| \leq r^{-1}$. Now consider $\tilde{\psi} = \theta \psi \in \operatorname{Lip}(\Omega)$. Clearly,

$$\tilde{\psi} = 0 \text{ on } B, \qquad \tilde{\psi} = \psi \text{ on } \Omega \setminus B_2.$$

We have

$$\textstyle\int_\Omega |\nabla\tilde\psi|^2 dx = \int_\Omega |\nabla(\theta\psi)|^2 dx \quad \begin{aligned} &\le 2\int_\Omega |\nabla\psi|^2 dx + 2r^{-2}\int_{B_2\setminus B}\psi^2 dx \\ &\le 2(\,\mathrm{cap}\,(F)+\varepsilon) + 2r^{-2}\int_{B_2}\psi^2 dx. \end{aligned}$$

On the other hand $\tilde\psi = 1$ near $\partial\Omega$, so if we take $\tilde u = 1-\tilde\psi$, then $\tilde u = 1$ on B and 0 near $\partial\Omega$, so $\tilde u$ can serve as a test function for estimating $\,\mathrm{cap}\,(\bar B)$. Therefore

$$\int_\Omega |\nabla\tilde\psi|^2 dx \ge \,\mathrm{cap}\,(\bar B).$$

Combining this inequality with the previous estimate we obtain

$$\mathrm{cap}\,(\bar B) \le 2(\,\mathrm{cap}\,(F)+\varepsilon) + 2r^{-2}\int_{B_2}\psi^2 dx,$$

hence

$$\int_{B_2}\psi^2 dx \ge \frac{1}{2}r^2\,\mathrm{cap}\,(\bar B) - r^2(\,\mathrm{cap}\,(F)+\varepsilon).$$

Now assuming that

$$\mathrm{cap}\,(F) \le \frac{1}{4}\,\mathrm{cap}\,\bar B - \varepsilon, \tag{4.5}$$

we obtain

$$\int_{B_2}\psi^2 dx \ge \frac{1}{4}r^2\,\mathrm{cap}\,(\bar B). \tag{4.6}$$

C. Now we should estimate the norm $\|\psi\|^2_{L^2(B)}$ from below. To this end let us use the estimate (2.9) with $t = 1/2$ and with r replaced by $2r$, i.e.

$$\|\psi\|^2_{L^2(B_2)} \le 2^{n+1}\|\psi\|^2_{L^2(B)} + 2^{2n+3}r^2\|\nabla\psi\|^2_{L^2(B_2)}.$$

Using the same ψ as above, we obtain

$$\|\nabla\psi\|^2_{L^2(B_2)} \le \|\nabla\psi\|^2_{L^2(\Omega)} \le \,\mathrm{cap}\,(F)+\varepsilon,$$

hence

$$\|\psi\|^2_{L^2(B)} \ge 2^{-n-1}\|\psi\|^2_{L^2(B_2)} - 2^{n+2}r^2\|\nabla\psi\|^2_{L^2(B_2)}$$
$$\ge 2^{-n-1}\|\psi\|^2_{L^2(B_2)} - 2^{n+2}r^2\|\nabla\psi\|^2_{L^2(\Omega)} \ge 2^{-n-1}\|\psi\|^2_{L^2(B_2)} - 2^{n+2}r^2(\,\mathrm{cap}\,(F)+\varepsilon).$$

Together with (4.6) this gives

$$\|\psi\|^2_{L^2(B)} \ge 2^{-n-3}r^2\,\mathrm{cap}\,(\bar B) - 2^{n+2}r^2(\,\mathrm{cap}\,(F)+\varepsilon),$$

provided (4.5) is satisfied.

Now imposing the condition

$$\operatorname{cap}(F) \le 2^{-2n-6} \operatorname{cap} \bar{B} - \varepsilon$$

which is obviously stronger than (4.5), we arrive to the estimate

$$\int_B \psi^2 dx \ge 2^{-n-4} r^2 \operatorname{cap}(\bar{B}).$$

Together with the inequality (4.4) this gives the estimate

$$\mu(B) \le 2^{n+4} r^{-2} [\operatorname{cap}(\bar{B})]^{-1} \left(\int_{B \setminus F} V(x) dx + \operatorname{cap}(F) + \varepsilon \right).$$

This estimate holds for any compact $F \subset B$ satisfying (4.2), with an arbitrary $\varepsilon > 0$ such that

$$\varepsilon < 2^{-2n-6} \operatorname{cap}(\bar{B}) - \operatorname{cap}(F).$$

Taking the limit as $\varepsilon \to 0$, we arrive to (4.1). □

Corollary 4.2. *There exists $c > 0$ such that the condition (D) in Theorem 1.1 is necessary i.e. $\sigma = \sigma_d$ implies (D).*

Proof. If (D) is not satisfied whatever $c > 0$, then Theorem 4.1 implies that $\mu(B(x,r))$ does not go to ∞ as $x \to \infty$. But then due to the localization theorem (Theorem 2.10) the spectrum of H is not discrete. □

5. Sufficient condition

In this section we will prove that the condition (D) in Theorem 1.1 is sufficient for the spectrum of the Schrödinger operator (1.1) to be discrete, i.e. (D) implies that $\sigma = \sigma_d$. As in Sect.2 we will also assume that $V \ge 1$. Using the localization theorem 2.10 and the bounded geometry conditions we could again argue in an Euclidean ball $B = B(0,r) \subset \mathbb{R}^n$ of sufficiently small radius $r > 0$. But it proves to be more convenient to use cubes instead of balls. Namely, we will use *geodesic cubic cells* which we will call simply *cells* i.e. sets which are given in geodesic coordinates as follows:

$$C(a,d) = \{x = (x^1, \dots, x^n) |\ |x^j - a^j| < d/2,\ j = 1, \dots, n\}.$$

Here $d > 0$ will be called the *size* of the cell, the point $a \in M$ is called the *center* of the cell, and all the points $x = (x^1, \dots, x^n) \in M$ are supposed to belong to a ball $B(x_0, R)$ where the geodesic coordinates $x^1, \dots, x^n$ are defined. We restrict ourselves to balls of sufficiently small radius $R < r_0$ with r_0 as in Sect.1. So by definition the cell $C(a, d)$ belongs to such a ball and working with this cell we will assume that the geodesic coordinates are fixed.

The closure of the cell $C(a, d)$ will be denoted $\bar{C}(a, d)$.

The advantage of using cells (instead of balls) is that we can subdivide cells into smaller cells up to sets of measure 0.

In the future we will mostly work in a fixed small cell $C = C(0, d) \subset \mathbb{R}^n$, i.e. the cell of the size d centered at the origin in $\mathbb{R}^n$. Here $\mathbb{R}^n$ is considered with the standard metric. The closure of this cell will be denoted $\bar{C}$.

Let us start with the following well known

Lemma 5.1. (Poincaré inequality)
Let $\bar{v}$ denote the mean value of a real-valued function $v \in \operatorname{Lip}(C)$ i.e.

$$\bar{v} = \frac{1}{\operatorname{vol}(C)} \int_C v(x)dx = \frac{1}{d^n} \int_C v(x)dx.$$

Then

$$\int_C (v - \bar{v})^2 dx \le \pi^{-2} d^2 \int_C |\nabla v|^2 dx.$$

Proof. Note that by separation of variables the first non-zero Neumann eigenvalue of $-\Delta$ in the cell $C = C(0, d) \subset \mathbb{R}^n$ is $\mu_1 = \pi^2 d^{-2}$.

Now the desired inequality follows from the standard variational formula for μ_1:

$$\mu_1 = \inf \left\{ \frac{\int_C |\nabla u|^2 dx}{\int_C |u|^2 dx} \,\middle|\, u \in \operatorname{Lip}(C), \int_C u dx = 0 \right\}.$$

$\square$

Let us introduce a class of *negligible* (or, more precisely, *c-negligible*) subsets in an open set $G \subset M$, such that G lies in a domain of geodesic coordinates, more precisely $G \subset B(x, R)$ for some $x \in M$ and $R < r_0/2$ where r_0 is chosen as in Sect.1. Using geodesic coordinates we can identify G with a subset in $\mathbb{R}^n$. Then we define

$$\mathcal{N}_c(G) = \{F \mid F \text{ is a compact subset in } \bar{G} \text{ and } \operatorname{cap}(F) \le c \operatorname{cap}(\bar{G})\}.$$

Here $c > 0$ is a constant, and cap means $\operatorname{cap}_\Omega$ where $\Omega = B(0, R) \subset \mathbb{R}^n$ as in the previous section, $R \in (0, r_0/2)$ is fixed. The class $\mathcal{N}_c(G)$ depends on G and also on the choice of the constant c which will be eventually chosen sufficiently small.

For the future we will only use G which is a small ball or a small cell i.e. $G = B$ or $G = C$, so we will assume that the boundary of G is piecewise smooth.

Note that it only makes sense to consider $\mathcal{N}_c(G)$ with $c < 1$ because otherwise this class consists of *all* compact subsets in G.

We will also need another class of negligible sets which depends on $\mu(G)$:

$$\mathcal{N}'_c(G) = \mathcal{N}_{c\mu(G)}(G).$$

Note that $\mu(G) \geq 1$ because $V \geq 1$. Therefore $\mathcal{N}_c(G) \subset \mathcal{N}'_c(G)$.

The main necessity Theorem 4.1 can be reformulated as the following estimate:

$$\mu(B) \leq A(r) \inf_{F \in \mathcal{N}_c(B)} \left(\int_{B \setminus F} V(x)dx + \operatorname{cap}(F) \right),$$

where $A(r) = 2^{n+5} r^{-2} [\operatorname{cap}(\bar{B})]^{-1}$, $c = 2^{-2n-7}$.

Our main goal in this section will be the proof of a somewhat opposite estimate which is formulated in the following

Theorem 5.2. *For any* $c > 0$ *there exists* $d_1 > 0$ *such that*

$$\inf_{F \in \mathcal{N}'_c(C)} \int_{C \setminus F} V(x)dx \leq 4d^n \mu(C), \tag{5.1}$$

provided $0 < d < d_1$.

In fact instead of (5.1) we will prove a stronger estimate which makes use of a more restrictive notion of negligibility. For any $A > 0$ define

$$\tilde{\mathcal{N}}_A(C) = \{F \mid F \text{ is a compact subset in } \bar{C} \text{ and } \operatorname{cap}(F) \leq Ad^n\}.$$

Similarly introduce also the corresponding class depending on $\mu(C)$:

$$\tilde{\mathcal{N}}'_A(C) = \tilde{\mathcal{N}}_{A\mu(C)}(C).$$

Again we have $\tilde{\mathcal{N}}_A(C) \subset \tilde{\mathcal{N}}'_A(C)$.

The following inclusions of subsets in $\mathbb{R}^n$

$$\bar{B}(0, d/2) \subset \bar{C}(0, d) \subset \bar{B}(0, d\sqrt{n}/2)$$

imply that

$$\operatorname{cap}(\bar{B}(0, d/2)) \leq \operatorname{cap}(\bar{C}(0, d)) \leq \operatorname{cap}(\bar{B}(0, d\sqrt{n}/2)).$$

Hence due to (3.13), (3.14) we see that

$$\operatorname{cap}(\bar{C}(0, d)) \asymp \operatorname{cap}(\bar{B}(0, d)). \tag{5.2}$$

(Let us recall that for two real-valued functions f_1, f_2 defined on the same set S, the relation $f_1 \asymp f_2$ means that there exists a constant $C > 0$ such that $C^{-1} f_1(x) \leq f_2(x) \leq C f_1(x)$ for all $x \in S$.)

Using (5.2) together with (3.13), (3.14), we easily conclude that for any $A > 0$ and $c > 0$ there exists $d_1 > 0$ such that

$$\tilde{\mathcal{N}}_A(C) \subset \mathcal{N}_c(C) \text{ and } \tilde{\mathcal{N}}'_A(C) \subset \mathcal{N}'_c(C) \text{ if } 0 < d < d_1,$$

because $d^n = o(\operatorname{cap}(\bar{C}))$ as $d \to 0$.

Proposition 5.3. *There exists* $A = A_n > 0$ *such that*

$$\inf_{F \in \tilde{\mathcal{N}}'_A(C)} \int_{C \setminus F} V(x)dx \le 4d^n \mu(C). \tag{5.3}$$

Since for small d the left-hand side of (5.3) is not smaller than the left-hand side of (5.1), Proposition 5.3 implies Theorem 5.2.

Proof of Proposition 5.3. A. Let us choose an arbitrary $\varepsilon > 0$ and take a real-valued function $\psi \in C^\infty(\bar{C})$ which nearly minimizes the fraction in (2.7), i.e. such that

$$\int_C (|\nabla\psi|^2 + V\psi^2)dx \le (1+\varepsilon)\mu(C) \int_C \psi^2 dx.$$

Let us normalize ψ so that $\overline{\psi^2} = 1$, i.e. $\|\psi\|^2_{L^2(C)} = \operatorname{vol}(C) = d^n$. We will have then

$$\int_C (|\nabla\psi|^2 + V\psi^2)dx \le (1+\varepsilon)d^n \mu(C). \tag{5.4}$$

Now let us take

$$F = \{x|\ x \in \bar{C},\ |\psi(x)| \le 1/2\}. \tag{5.5}$$

Then

$$\int_{C\setminus F} V dx \le 4\int_{C\setminus F} V\psi^2 dx \le 4\int_C (|\nabla\psi|^2 + V\psi^2)dx \le 4d^n(1+\varepsilon)\mu(C).$$

After taking infimum over all $F \in \mathcal{N}'_A(C)$ in the left-hand side here, we can replace ε by 0 in the right-hand side provided we can choose $A = A_n$ which does not depend on ε (and on d). Then the desired estimate (5.3) will follow.

B. Now we have to estimate the capacity of the set F defined by (5.5). Using the same function ψ as above, let us consider $\tilde{\psi} \in \operatorname{Lip}(C)$ defined as follows:

$$\tilde{\psi}(x) = \max(|\psi(x)| - \frac{1}{2}, 0).$$

Clearly $|\tilde{\psi}| \le |\psi| \le |\tilde{\psi}| + 1/2$ on C and $\tilde{\psi} = 0$ on F. Also $|\nabla\tilde{\psi}(x)| \le |\nabla\psi(x)|$ for almost all $x \in C$, hence

$$\int_C |\nabla\tilde{\psi}|^2 dx \le \int_C |\nabla\psi|^2 dx. \tag{5.6}$$

We also have

$$d^n = \int_C \psi^2 dx \le \int_C \left(|\tilde{\psi}| + \frac{1}{2} \right)^2 dx \le 2 \int_C |\tilde{\psi}|^2 dx + \frac{1}{2} d^n.$$

Therefore

$$\int_C \tilde{\psi}^2 dx \ge \frac{1}{4} d^n. \tag{5.7}$$

Normalize $\tilde{\psi}$ in the same way as ψ i.e. take $\phi = a\tilde{\psi}$ with $a > 0$ so that $\overline{\phi^2} = 1$. Then it follows from (5.7) that we can take $1 \le a \le 4$, hence (5.6) and (5.4) imply that

$$\int_C |\nabla \phi|^2 dx \le 4(1+\varepsilon) d^n \mu(C). \tag{5.8}$$

Note that $\phi = 0$ on F.

We will also assume (replacing ϕ by $-\phi$ if necessary) that $\overline{\phi} \ge 0$.

C. Now define $v = 1 - \phi$, so $v = 1$ on F. Let us extend v to $C_2 = C(0, 2d)$ by reflections in hyperplanes constituting the faces of the cube $C = C(0, d)$. Namely, we may e.g. first use reflections in the planes $x^1 = \pm d/2$ to extend v to the set $(-d, d) \times (-d/2, d/2)^{n-1}$, then use reflections in the planes $x^2 = \pm d/2$ to extend the resulting function to the set $(-d, d)^2 \times (-d/2, d/2)^{n-2}$ etc. The result will be a function $\hat{v} \in \mathrm{Lip}\,(C_2)$ such that

$$\int_{C_2} \hat{v}^2 dx = 2^n \int_C v^2 dx, \tag{5.9}$$

and

$$\int_{C_2} |\nabla \hat{v}|^2 dx = 2^n \int_C |\nabla v|^2 dx.$$

Since $|\nabla v| = |\nabla \phi|$ on C, we obtain also according to (5.8):

$$\int_{C_2} |\nabla \hat{v}|^2 dx \le 2^{n+2} d^n (1+\varepsilon) \mu(C). \tag{5.10}$$

For the simplicity of notations we will write further v instead of $\hat{v}$ which obviously does not lead to a confusion.

D. To get a test function for estimating the capacity of F, we need to cut-off v to obtain a Lipschitz function in Ω such that it vanishes at $\partial\Omega$. In fact we will even make it vanish outside C_2. To this end let us take a cut-off function $\chi \in \mathrm{Lip}\,(C_2)$ such that $\chi = 1$ on C, $\chi = 0$ on ∂C_2, $0 \le \chi \le 1$ everywhere and $|\nabla \chi| \le 2\sqrt{n} d^{-1}$. (We can take e.g. $\chi(x) = \chi_0(x^1)\chi_0(x^2) \dots \chi_0(x^n)$ where $\chi_0(t) = 1$ on $[-d/2, d/2]$, 0 on $\mathbb{R} \setminus (-d, d)$ and χ_0 is linear on each of the intervals $[-d, -d/2]$ and $[d/2, d]$.)

Define $u = \chi v \in \mathrm{Lip}\,(C_2)$, so $u = 1$ on F and $u = 0$ on ∂C_2. We can now extend u by 0 to $\Omega \setminus \bar{C}_2$.

Let us estimate the Dirichlet integral for u. We have

$$(5.11) \int_{C_2} |\nabla u|^2 dx = \int_{C_2} |\chi \nabla v + v \nabla \chi|^2 dx \le 2 \int_{C_2} |\nabla v|^2 dx + 8nd^{-2} \int_{C_2} v^2 dx.$$

E. Our next task will be to prove that $\|v\|^2_{L^2(C_2)}$ is small, so that we may in fact omit the last term from the estimate (5.11), installing an additional constant factor in front of $\|\nabla v\|^2_{L^2(C_2)}$ instead. The idea is that the L^2-norm of $\nabla \phi$ is small, therefore ϕ is "almost constant", hence close to 1. We will prove that this is indeed true in the L^2 sense. (The proof makes use of the Poincaré inequality.) It will follow that $v = 1 - \phi$ is close to 0 in the same sense.

Note that (5.9) shows that it is sufficient to evaluate $\|v\|^2_{L^2(C)}$.

Our first step will be to show that $\bar{v} = 1 - \bar{\phi}$ is small, more precisely:

$$(5.12) \qquad |\bar{\phi} - 1| \le 2\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d.$$

Using the triangle inequality and Lemma 5.1 we obtain

$$d^{n/2} = \|\phi\| \le \|\phi - \bar{\phi}\| + \bar{\phi} d^{n/2} \le \pi^{-1} d \|\nabla \phi\| + \bar{\phi} d^{n/2},$$

where $\|\cdot\|$ means the norm in $L^2(C)$. Now using (5.8) we obtain

$$d^{n/2} \le 2\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d^{n/2+1} + \bar{\phi} d^{n/2},$$

or

$$(5.13) \qquad \bar{\phi} \ge 1 - 2\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d.$$

Similarly we have

$$\begin{aligned} \bar{\phi} d^{n/2} = \|\bar{\phi}\| &\le \|\phi - \bar{\phi}\| + \|\phi\| \le \pi^{-1} d \|\nabla \phi\| + d^{n/2} \\ &\le 2\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d^{n/2+1} + d^{n/2}, \end{aligned}$$

hence

$$(5.14) \qquad \bar{\phi} \le 1 + 2\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d.$$

Combining (5.13) and (5.14) we obtain (5.12).

F. Clearly (5.12) can be rewritten as

$$|\bar{v}| \le 2\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d.$$

Now arguing as in the proof of (5.12) we can write

$$\begin{aligned} \|v\| &\le \|\bar{v}\| + \|v - \bar{v}\| \le 2\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d^{n/2+1} + \pi^{-1} d \|\nabla v\| \\ &\le 2\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d^{n/2+1} + 2\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d^{n/2+1} \\ &= 4\pi^{-1}[(1+\varepsilon)\mu(C)]^{1/2} d^{n/2+1}, \end{aligned}$$

so we finally obtain the desired estimate

$$\int_C v^2 dx \le 16\pi^{-2}(1+\varepsilon)\mu(C)d^{n+2},$$

which implies due to (5.9):

$$\int_{C_2} v^2 dx \le 2^{n+4}\pi^{-2}(1+\varepsilon)\mu(C)d^{n+2}. \tag{5.15}$$

G. Now let us return to estimating u. We can use (5.11), (5.10) and (5.15) to conclude that

$$\operatorname{cap}(F) \le A_n \mu(C) d^n,$$

where $A_n = 2^{n+3}(1+\varepsilon)(1+16n\pi^{-2})$. This concludes the proof of Proposition 5.3. $\square$

Proof of Theorem 1.1. A. We already proved in Sect.4 that the condition (D) is necessary. To prove that it is also sufficient, let us assume the opposite i.e. that (D) is satisfied but $\sigma \ne \sigma_d$. It follows due to the localization theorem 2.10 that there exist small $r > 0$ and a sequence of balls $B(x_k, r)$, $k = 1, 2, \ldots$, such that $x_k \to \infty$ as $k \to \infty$ but $\lambda(B(x_k, r)) \le L$ where $L > 0$ does not depend on k. Using the geodesic coordinates centered at x_k and the monotonicity of $\lambda(G)$ with respect to the ordering of G by inclusion we deduce that $\lambda(C(x_k, 2r)) \le L$ with the same constant L because $B(x_k, r) \subset C(x_k, 2r)$. It follows that

$$\mu(C(x_k, d)) \le L, \tag{5.16}$$

where $d = 2r$.

B. Fixing k for a moment and choosing an arbitrary integer $N > 0$ we can in a standard way subdivide the cell $C = C(x_k, d)$ into N^n subcells

$$C_j = C(x'_{kj}, d/N), \qquad j = 1, \ldots, N^n,$$

so that the closures of the subcells C_j cover exactly the closure of C and the interiors of these subcells do not intersect. We claim that then there exists $j \in \{1, \ldots, N^n\}$ such that

$$\mu(C_j) \le L + 2,$$

with the same L as in (5.16).

Indeed, assuming opposite, we conclude that for any j the inequality

$$\int_{C_j} (|\nabla\psi|^2 + V|\psi|^2) d\mu \ge (L+1) \int_{C_j} |\psi|^2 d\mu$$

holds for any function $\psi \in C^\infty(C_j)$. But then we can take an arbitrary $\psi \in C^\infty(C)$ and sum up the inequalities above, applying them to the restrictions of ψ to the subcells C_j. In this way we see that the inequality

$$\int_C (|\nabla\psi|^2 + V|\psi|^2)d\mu \geq (L+1)\int_C |\psi|^2 d\mu$$

holds for any $\psi \in C^\infty(C)$. This means that $\mu(C) \geq L+1$, which contradicts (5.16).

We conclude therefore that for any integer $N > 0$ there exists a sequence of points $\{x_k^{(N)} | \ k = 1, 2, \ldots\}$ in M such that $x_k^{(N)} \to \infty$ as $k \to \infty$ for any fixed N, but $\mu(C(x_k^{(N)}, d/N)) \leq L+1$ with a constant L which is independent of N.

C. Let us denote temporarily $\tilde{C} = C(x_k^{(N)}, d/N)$. We can use Theorem 5.2 to deduce that for any $c > 0$ there exists $N > 0$ such that in geodesic coordinates

$$\inf_{F \in \mathcal{N}_c(\tilde{C})} \int_{\tilde{C} \setminus F} V(x)dx \leq 4\tilde{L}N^{-n}d^n,$$

where $\tilde{L} > 0$ does not depend on k and N. Let us show that the same will be true if we replace the cell $\tilde{C}$ by the ball $\tilde{B} = B(x_k^{(N)}, d/2N) \subset \tilde{C}$, which clearly leads to a contradiction with the condition (D) and ends the proof of Theorem 1.1 (we should take $c > 0$ sufficiently small so that the condition (D) is also necessary).

D. To be able to replace $\tilde{C}$ by $\tilde{B}$ it is sufficient to establish that for any $c > 0$ there exists $b > 0$ such that

$$\inf_{F_1 \in \mathcal{N}_c(\tilde{C})} \int_{\tilde{C} \setminus F_1} V(x)dx \geq \inf_{F_2 \in \mathcal{N}_b(\tilde{B})} \int_{\tilde{B} \setminus F_2} V(x)dx. \tag{5.17}$$

To this end note first that for any compact set $F_1 \subset \tilde{C}$ we can take $F_2 = F_1 \cap \overline{\tilde{B}}$ to get the inequality

$$\int_{\tilde{C} \setminus F_1} V(x)dx \geq \int_{\tilde{B} \setminus F_2} V(x)dx.$$

This would be enough to establish (5.17) if F_2 would be a compact subset in $\tilde{B}$. But we can now replace F_2 by an intersection of F_2 with a smaller ball, e.g. by $F_2^{(m)} = F_2 \cap B(x_k^{(N)}, d/2N - 1/m)$ and use the obvious relation

$$\int_{\tilde{B} \setminus F_2} V(x)dx = \lim_{m \to \infty} \int_{\tilde{B} \setminus F_2^{(m)}} V(x)dx.$$

This leads to (5.17) and ends the proof of Theorem 1.1. □

Remark 5.4. The arguments given in the proof of Theorem 1.1 show that the condition (D) for balls is equivalent to the similar condition for geodesic cubic cells.

Another strange corollary of the arguments above is the fact that the negligibility restriction $\operatorname{cap}(F) \le c \operatorname{cap}(\bar{B})$ can be replaced by a formally much stronger restriction $\operatorname{cap}(F) \le Ar^n$ (here we should take arbitrary $A > 0$ but then introduce restriction $r \in (0, r_1)$ where $r_1 = r_1(A)$). The resulting version $(D)_s$ of the condition (D) is equivalent to the original condition. Indeed, we actually proved that $(D)_s$ implies that $\sigma = \sigma_d$ which in turn implies (D), but also obviously (D) implies $(D)_s$, hence (D) and $(D)_s$ are equivalent.

6. Applications, examples and further results

6.1. Measure conditions

We will start by proving that replacing the capacity by the Lebesgue measure in the condition (D) leads to a sufficient condition of the discreteness of the spectrum.

Theorem 6.1. *1) Assume that (M, g) is a Riemannian manifold of bounded geometry, H is the Schrödinger operator (1.1) with the potential satisfying (1.2).*

Let us assume that there exists $c > 0$ such that the following condition is satisfied:

(D_L) For any sequence $\{x_k |\ k = 1, 2, \ldots\} \subset M$ such that $x_k \to \infty$ as $k \to \infty$, for any $r < r_0/2$ and any compact subsets $F_k \subset B(x_k, r)$ with $\operatorname{mes} F_k \le cr^n$ we have

$$\int_{B(x_k,r)\setminus F_k} V(x) d\mu(x) \to \infty \quad \textit{as} \quad k \to \infty . \tag{6.1}$$

Then $\sigma = \sigma_d$.

Here $\operatorname{mes} F_k$ means the Riemannian measure of the set F_k or its Lebesgue measure in the normal (geodesic) coordinates centered at x_k.

2) If $n = 2$, then a stronger statement also holds. Namely, assume that there exist $N > 0$, $c > 0$ and $r_1 > 0$ such that the condition (6.1) holds provided $r \in (0, r_1)$ and $\operatorname{mes} F_k \le cr^N$. Then $\sigma = \sigma_d$.

Proof. We will use the following inequalities comparing the capacity with the Lebesgue measure in $\mathbb{R}^n$ ([55], Sect.2.2.3):

$$\operatorname{cap}_\Omega(F) \ge \omega_n^{2/n} n^{(n-2)/n} (n-2) (\operatorname{mes} F)^{(n-2)/n}, \qquad n \ge 3, \tag{6.2}$$

and

$$\operatorname{cap}_\Omega(F) \ge 4\pi \left[\log \left(\frac{\operatorname{mes} \Omega}{\operatorname{mes} F} \right) \right]^{-1}, \qquad n = 2. \tag{6.3}$$

Here $\Omega = B(0, R) \subset \mathbb{R}^n$ and $R > 0$ is fixed, F is a compact subset in Ω.

Now we will prove that (D_L) implies (D), possibly with a different constant c. For simplicity of notations we will write cap instead of $\operatorname{cap}_\Omega$.

Assume first that $n \geq 3$ and (D_L) is satisfied. This means that (6.1) is satisfied provided

$$\operatorname{mes} F_k \leq cr^n \tag{6.4}$$

with a constant $c > 0$. We need to prove that (6.1) is true under the condition

$$\operatorname{cap}(F_k) \leq \tilde{c} r^{n-2} \tag{6.5}$$

with another constant $\tilde{c} > 0$. To this end it is sufficient to prove that (6.5) with an appropriate choice of the constant $\tilde{c}$ implies (6.4). But this is obvious because $\operatorname{mes} F_k \leq A_n [\operatorname{cap}(F_k)]^{n/(n-2)}$ due to (6.2).

Arguing similarly in the case $n = 2$, we see that it is sufficient to prove that

$$\operatorname{cap}(F_k) \leq \tilde{c} \left(\log \frac{1}{r} \right)^{-1} \tag{6.6}$$

implies

$$\operatorname{mes} F_k \leq cr^N,$$

provided $r \in (0, r_1)$ and the constants $r_1, \tilde{c}$ are appropriately chosen. To establish this we can use (6.3) and (6.6) to conclude that

$$\begin{aligned} \operatorname{mes} F_k &\leq \operatorname{mes} \Omega \cdot \exp(-4\pi (\operatorname{cap}(F_k))^{-1}) \leq \operatorname{mes} \Omega \cdot \exp(-4\pi \tilde{c}^{-1} \log \frac{1}{r}) \\ &= \operatorname{mes} \Omega \cdot r^{4\pi \tilde{c}^{-1}} \leq cr^N, \end{aligned}$$

provided $r \in (0, r_1)$ and both $\tilde{c}, r_1$ are sufficiently small. □

Corollary 6.2. *Under the conditions of Theorem 1.1 assume that for any* $A > 0$ *and any* $r \in (0, r_0/2)$

$$\operatorname{mes} \{y |\ y \in B(x, r), V(y) \leq A\} \to 0 \quad \text{as} \quad x \to \infty. \tag{6.7}$$

Then $\sigma = \sigma_d$.

Proof. Let mes denote the Lebesgue measure in geodesic coordinates. Let us prove that (6.7) implies the condition (D_L) in Theorem 6.1. Indeed, we can obviously take c to be arbitrarily small. Then we will have

$$\operatorname{mes}(B(x_k, r)) \setminus F_k) \geq \frac{1}{2} \operatorname{mes} B(x_k, r).$$

It follows that for any $A > 0$ and large $k \geq k_A$ we have

$$\operatorname{mes}(\{y |\ V(y) \geq A\} \cap (B(x_k, r) \setminus F_k)) \geq \frac{1}{4} \operatorname{mes} B(x_k, r),$$

hence

$$\int_{B(x_k,r)\setminus F_k} V(x)dx \geq \frac{1}{4} A \operatorname{mes} B(x_k, r),$$

which implies (6.1). □

Example 6.3. For $n = 2$ in $\mathbb{R}^2$ with the standard metric consider the Schrödinger operator $H = -\Delta + x^2y^2$ where $x = x^1, y = x^2$. It is easy to see that the condition (6.7) is satisfied. (The reason is that the set $\{(x, y) |\, x^2y^2 \leq A\}$ becomes narrow at infinity, so its intersections with a ball of a fixed radius r have the measure which tends to 0 as the ball goes to infinity.) Therefore $\sigma = \sigma_d$.

Similarly we can use Corollary 6.2 to establish that the operator

$$H = -\Delta + x^2y^2 + x^2z^2 + y^2z^2$$

in $\mathbb{R}^3 = \{(x, y, z)\}$ has discrete spectrum.

6.2. Dirichlet spectrum in open subsets

Let us start with some generalities. Let (M, g) be a Riemannian manifold with bounded geometry. Let us choose an open subset $G \subset M$ and consider the Schrödinger operator (1.1) with the potential $V \in L^1_{loc}(G)$, $V \geq 0$, and with the Dirichlet boundary condition. It is defined as the self-adjoint operator H_G which is canonically associated with the closure of the quadratic form (2.1) defined on $C_c^\infty(G)$. (See the beginning of Sect.2 for the definitions.)

It is easy to see that the form Q indeed admits the closure (see the arguments at the beginning of Sect.2).

A particular case is the Laplacian with the Dirichlet boundary condition (the case $V \equiv 0$). The corresponding operator will be denoted $-\Delta_G$. In this case the quadratic form is the Dirichlet integral

$$\int_M |\nabla u|^2 d\mu = \int_M g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \sqrt{g} dx$$

defined a priori on $C_c^\infty(G)$.

We can consider H_G as a particular case (or the limit case) of the Schrödinger operator $H = -\Delta + V(x)$ with $V = +\infty$ on $M \setminus G$ but a mechanical application of the formulations taken from the case $V \in L^1_{loc}(M)$ in fact gives wrong result. Some arguments should be modified, though not very much, so basically the proof of Theorem 1.1 works in this situation. The following theorem is an appropriate reformulation of Theorem 1.1 for this situation. In case $M = \mathbb{R}^n$ it is due to V.G. Maz'ya [55], Sect.12.5.

Theorem 6.4. *There exists $c = c(M, g) > 0$ such that the spectrum of H_G is discrete if and only if the following condition is fulfilled:*

(D_G) Assume that we are given a sequence $\{x_k|\ k=1,2,\dots\}\subset M$ such that $x_k\to\infty$ as $k\to\infty$, a number $r\in(0,r_0/2)$ and compact subsets $F_k\subset\bar{B}(x_k,r)$ satisfying the conditions

$$F_k\supset(M\setminus G)\cap\bar{B}(x_k,r),$$

$\operatorname{cap}(F_k)\le cr^{n-2}$ *in case* $n\ge 3$ *and* $\operatorname{cap}(F_k)\le c\left(\log\frac{1}{r}\right)^{-1}$ *in case* $n=2$*. Then*

$$\int_{B(x_k,r)\setminus F_k} V(x)d\mu(x)\to\infty \quad \text{as} \quad k\to\infty.$$

An equivalent condition is obtained if we require that (D_G) is true for all sufficiently small $c>0$.

It can happen that there is no $\{x_k\}$, r and $\{F_k\}$ satisfying the conditions in (D_G). Then it is understood that the condition (D_G) is satisfied, hence $\sigma=\sigma_d$ for H_G for any potential $V\in L^1_{loc}(G)$, $V\ge 0$. This will be true if and only if $\sigma=\sigma_d$ for $-\Delta_G$. Let us formulate this result as the following

Corollary 6.5. *There exists $c=c(M,g)>0$ such that the spectrum of $-\Delta_G$ is discrete if and only if for any $r\in(0,r_0/2)$*

$$\liminf_{x\to\infty}\operatorname{cap}\left((M\setminus G)\cap\bar{B}(x,r)\right)\ge cr^{n-2}, \quad \text{if} \quad n\ge 3,$$

and

$$\liminf_{x\to\infty}\operatorname{cap}\left((M\setminus G)\cap\bar{B}(x,r)\right)\ge c\left(\log\frac{1}{r}\right)^{-1}, \quad \text{if} \quad n=2.$$

An equivalent condition is obtained if we require that these conditions hold for all sufficiently small $c>0$.

This Corollary in case $M=\mathbb{R}^n$ was first mentioned by A.M. Molchanov (see [59], Theorem 8.1).

Sketch of Proof of Theorem 6.4. A. We can assume without loss of generality that $V(x)\ge 1$ for all $x\in G$. Let us introduce the set

$$\mathcal{L}_G=\{u\in C_c^\infty(G)|\ \int_G(|\nabla u|^2+V|u^2|)d\mu\le 1\}.$$

Then $\sigma=\sigma_d$ for H_G if and only if $\mathcal{L}_G$ is precompact in $L^2(G)$. Clearly Lemmas 2.2 and 2.3 hold in this situation.

Now for any open set $\Omega\subset M$ we should introduce the numbers

$$\lambda_G(\Omega)=\inf\left\{\frac{\int_\Omega(|\nabla u|^2+V|u|^2)d\mu}{\int_\Omega|u|^2d\mu},\ u\in C_c^\infty(\Omega\cap G)\setminus 0\right\}, \tag{6.8}$$

$$\mu_G(\Omega) = \inf\left\{ \frac{\int_\Omega (|\nabla u|^2 + V|u|^2)d\mu}{\int_\Omega |u|^2 d\mu},\ u \in C_c^\infty(G)\setminus 0 \right\}\ . \tag{6.9}$$

Clearly $\lambda_G(\Omega)$ is the bottom of the Dirichlet spectrum of H in $\Omega \cap G$, and $\mu_G(\Omega)$ is the bottom of the spectrum of H in $\Omega \cap G$ with the Dirichlet condition on $\partial G \cap \partial(\Omega \cap G)$ and the Neumann condition on the remaining part of ∂G (which is contained in $\partial\Omega$). Then it is easy to establish the localization result which is obtained from Theorem 2.10 by replacing λ and μ by λ_G and μ_G respectively. To this end we can use Lemma 2.8 and note that all arguments from the proofs of Proposition 2.4, Proposition 2.7, Lemma 2.9 and Theorem 2.10 work without any changes if we restrict ourselves to functions which vanish near ∂G.

B. We can extend Theorem 4.1 to evaluate $\mu_G(B)$ where G is an open subset in $\mathbb{R}^n$, if we require that $F \supset (\mathbb{R}^n \setminus G) \cap \bar{B}$. (Of course it is possible that we will not be able to find such a set F which satisfies the capacity restriction (4.2) but otherwise the proof works without any changes and provides the necessity of the condition D_G.

C. The arguments of Sect.5 work as well towards the proof of sufficiency of (D_G). Again we should work with $\mu_G(C)$ instead of $\mu(C)$ and start with a function $\psi \in C_c^\infty(G)$ which nearly minimizes the ratio in (6.9). Then for the set F defined by (5.5) we have

$$F \supset (M \setminus G) \cap \bar{C},$$

and the rest of the proof of sufficiency does not require any changes. □

Remark 6.6. Note that if we try to apply formally Theorem 1.1 in the situation of Theorem 6.4, by extending V by $+\infty$ on $M \setminus G$, then we arrive to a condition which is not equivalent to (D_G) even for $V \equiv 0$, hence we get wrong result even for $-\Delta_G$.

Again we can use the comparison between the capacity and the Lebesgue measure to obtain a sufficient condition which is similar to Theorem 6.1:

Theorem 6.7. *1) Let us assume that there exists $c > 0$ such that the following condition is satisfied:*

(D_{GL}) *For any sequence $\{x_k|\ k = 1, 2, \ldots\} \subset M$ such that $x_k \to \infty$ as $k \to \infty$, for any $r < r_0/2$ and any compact subsets $F_k \subset \bar{B}(x_k, r)$ with $F_k \supset (M \setminus G) \cap \bar{B}(x_k, r)$ and $\operatorname{mes} F_k \le cr^n$ we have*

$$\int_{B(x_k,r)\setminus F_k} V(x)d\mu(x) \to \infty \quad as \quad k \to \infty\ . \tag{6.10}$$

Then H_G has a discrete spectrum.

2) If $n = 2$, then a stronger statement also holds. Namely, assume that there exist $N > 0$, $c > 0$ and $r_1 > 0$ such that the condition (6.10) holds provided

$r \in (0, r_1)$, $F_k \supset (M \setminus G) \cap \bar{B}(x_k, r)$ and $\operatorname{mes} F_k \leq cr^N$. Then H_G has a discrete spectrum.

For the particular case $V \equiv 0$ we get the following

Corollary 6.8. *1) Assume that there exist $c > 0$ and $r_1 > 0$ such that for any $r \in (0, r_1)$*

$$\liminf_{x\to\infty} \operatorname{mes}\left((M \setminus G) \cap \bar{B}(x, r)\right) \geq cr^n.$$

Then $-\Delta_G$ has a discrete spectrum.

2) Assume that $n = 2$ and there exist $c > 0$, $N > 0$ and $r_1 > 0$ such that for any $r \in (0, r_1)$

$$\liminf_{x\to\infty} \operatorname{mes}\left((M \setminus G) \cap \bar{B}(x, r)\right) \geq cr^N.$$

Then $-\Delta_G$ has a discrete spectrum.

In particular, we have

Corollary 6.9. *Assume that for any fixed $r \in (0, r_1)$*

$$\operatorname{mes}\left(G \cap B(x, r)\right) \to 0 \quad \text{as} \quad x \to \infty.$$

Then $-\Delta_G$ has a discrete spectrum.

This in turns implies

Corollary 6.10. *If $\operatorname{vol}(G) < \infty$, then $-\Delta_G$ has a discrete spectrum.*

The condition that G is an open set in a manifold of bounded geometry plays an important role here. For example, it is well known that non-compact hyperbolic surfaces of finite volume never have a discrete spectrum. Also it is well known that even in a bounded open set $G \subset \mathbb{R}^n$ the Laplacian $-\Delta$ with the Neumann boundary condition may have non-discrete spectrum.

Of course there are plenty of open sets G with infinite volume and with a discrete spectrum of $-\Delta_G$. The following interesting example is well known.

Example 6.11. We will describe now "the spiny urchin" which is a connected open subset $G \subset \mathbb{R}^2$ introduced by C. Clark [20]. It is obtained by removing countably many disjoint rays from $\mathbb{R}^2$. Namely, let us identify $\mathbb{R}^2 = \mathbb{C}$ and take

$$G = \mathbb{R}^2 \setminus \bigcup_{k=1}^{\infty} \{z|\ \arg z = n\pi 2^{-k} \text{ for some integer } n,\ |z| \geq k\}.$$

Any circle $\{z|\ |z| = a\}$ with $k - 1 < a < k$ is divided by the rays into 2^k equal parts. Therefore it is clear that if we take a closed disc $\bar{B}(z, r) \subset \mathbb{C}$ with a fixed $r > 0$ and $z \to \infty$, then for large $|z|$ this disc will be uniformly densely pierced by a number of rays which goes to infinity as $|z| \to \infty$. It can be deduced from this geometric picture that $\operatorname{cap}(F \cap \bar{B}(z,r)) \to \operatorname{cap}(\bar{B}(z,r))$ as $z \to \infty$. (Here we can assume that $r < 1$ and define the capacity of $\operatorname{cap}(F \cap \bar{B}(z,r))$ as the capacity with respect to $B(z,1)$.) Therefore Corollary 6.5 (but not Corollary 6.8) allows us to conclude that for the open set G described above the Laplacian $-\Delta_G$ has a discrete spectrum. as was first established in [20].

C. Clark [20] in fact established the discreteness of the spectrum for $-\Delta_G$ by elementary methods. We can also deduce it directly from the G-version of the localization theorem 2.10 (see the proof of Theorem 6.4). Indeed, $\lambda_G(B(z,r)) = \lambda(G \cap B(z,r))$ is equal to the minimum Dirichlet eigenvalue for all connected components of $G \cap B(z,r)$. But these components become more and more narrow as $z \to \infty$ i.e. their maximal width tends to 0 as $z \to \infty$. It easily follows that $\lambda(G \cap B(z,r)) \to \infty$ as $z \to \infty$, hence the spectrum of $-\Delta_G$ is discrete.

Of course more can be said about the spectrum of this particular domain. For example, J. Fleckinger ([28], [29]) obtained an asymptotic of the eigenvalues distribution function for the spiny urchin.

The following theorem generalizes Molchanov's Theorem 8.3 [59] and follows from the results by H. Donnelly [26] who only required lower bound for the Ricci curvature instead of bounded geometry.

Theorem 6.12. *Let (M,g) be a Riemannian manifold of bounded geometry, $G \subset M$ is an open set such that ∂G has bounded coefficients of the second quadratic form in M ("bounded curvature"). Then $-\Delta_G$ has a discrete spectrum if and only if for any fixed $r > 0$ the set G does not contain any infinite set of disjoint balls with the radius r.*

Sketch of the proof. Clearly the condition is necessary i.e. if there exists $r > 0$ such that G contains an infinite set of disjoint balls with the radius r, then $\sigma \neq \sigma_d$ (for $-\Delta_G$). Let us prove the inverse statement. Assume that $\sigma \neq \sigma_d$. Then by the Corollary 6.5 there exists an infinite set of disjoint balls $\bar{B}(x_k,r) \subset M$ which have "negligible" intersections with $M \setminus G$ i.e.

$$\operatorname{cap}((M \setminus G) \cap \bar{B}(x_k,r)) \leq c \operatorname{cap}(\bar{B}(x_k,r)) \tag{6.11}$$

with a small $c > 0$. The "bounded curvature" restriction on ∂G implies that there exists $r_1 > 0$ such that for any $x \in M$ and $r \in (0,r_1)$ the connected components of $\partial G \cap \bar{B}(x,r)$ either belong to a boundary layer $\bar{B}(x,r) \setminus B(x,\frac{1}{3}r)$ or are transversal to the sphere $\partial B(x,r)$ and behave as hyperplane sections of $\bar{B}(x,r)$ in geodesic coordinates. In particular if such a component intersects $B(x,\frac{1}{3}r)$ then it has the Riemannian $(n-1)$-measure which is comparable with r^{n-1}, hence due to an isoperimetric inequality (see e.g. Sect.2.3.3 in [55]) it has the capacity which is comparable with the capacity of the ball $\bar{B}(x,r)$.

So we have the following alternative: every connected component of the intersection $((M \setminus G) \cap \bar{B}(x,r))$ either is disjoint with the smaller ball $B(x,r/3)$ or has a capacity which is comparable with the capacity of the ball $B(x,r)$.

Applying this to the balls $B(x_k, r)$ chosen above (we can assume of course that $r < r_1$), we see that the second possibility is forbidden by (6.11), hence $B(x_k, r/3) \subset G$ for all k. ☐

In case $n = 2$ the "bounded curvature" restriction can be replaced by the requirement of the connectedness of $M \setminus G$ as it is seen in the following result generalizing another Molchanov's result (Theorem 8.2 in [59]):

Theorem 6.13. *Assume that (M, g) is a manifold of bounded geometry with $\dim M = 2$, $G \subset M$ is open and $M \setminus G$ is connected. Then the spectrum of $-\Delta_G$ is discrete if and only if for any fixed $r > 0$ the domain G does not contains any infinite set of disjoint discs with the radius r.*

Sketch of the proof. The proof is similar to the proof of Theorem 6.12.
Note that the connectedness of $M \setminus G$ implies that either $B(x, r/3) \subset G$ or the diameter of a connected component of $(M \setminus G) \cap \bar{B}(x, r)$ is at least $\frac{2}{3}r$.

In the last case it follows that $\operatorname{cap}((M \setminus G) \cap \bar{B}(x, r))$ is comparable with the capacity of the ball $\bar{B}(x, r)$ (see [55], Proposition 2 in Sect. 9.1.2). Hence we can again apply Corollary 6.5 to end the proof. ☐

Remark 6.14. Results similar to Theorems 6.12, 6.13 for the Neumann boundary condition must include some regularity conditions for the boundary of G, because it is well known that even bounded domains in $\mathbb{R}^n$ may have non-discrete Neumann spectrum (see e.g. [2], and Sect.XIII.14 in [64]). It is easy to see that if the Neumann spectrum of $-\Delta$ in a domain $G \subset \mathbb{R}^n$ is discrete, then $\operatorname{vol}(G) < \infty$. V.G. Maz'ya ([55], Sect.4.10) gave a necessary and sufficient condition (in terms of capacity) for the discreteness of the spectrum in this situation.

6.3. Review of some related results

The literature about the discreteness of spectrum starts with a paper by H. Weyl [79] who proved that the spectrum of the one-dimensional Schrödinger operator $H_1 = -d^2/dt^2 + V(t)$ in $L^2([0, +\infty))$ is discrete if $V(x)$ is monotone and $V(t) \to +\infty$ as $t \to +\infty$. The monotonicity requirement can be removed (see e.g. K. Friedrichs [30], and also books by E.C. Titchmarsh [74], I.M. Glazman [32], M. Reed and B. Simon [64], F.A. Berezin and M.A. Shubin [4]).

After the Molchanov's result [59] many improvements were made for the one-dimensional Schrödinger operator H_1 in $L^2(\mathbb{R})$. In particular, the semi-boundedness requirement was relaxed by I. Brinck [9] who proved that if we replace (1.2) by requiring only that

$$\int_J V(t)dx \geq -C \tag{6.12}$$

for any interval $J \subset \mathbb{R}$ of length ≤ 1, then Molchanov's condition (1.5) is still necessary and sufficient for the discreteness of the spectrum. Further improvements are due to R.S. Ismagilov [34] and L.B. Zelenko [81].

Another non-trivial one-dimensional result is due to I.S. Kac and M.G. Krein [35] who gave an explicit necessary and sufficient condition for the discreteness of spectrum of a singular string.

The one-dimensional Molchanov theorem and its above mentioned improvements were extended by several authors to the operators H_1 with operator-valued potentials V whose values are self-adjoint (not necessarily bounded) operators in a Hilbert space such that $\operatorname{Dom}(V(t))$ does not depend of t. One of the first results of this kind is due to B.M. Levitan and G.A. Suvorchenkova [44] who proved that if V is bounded below in operator sense, then the Molchanov condition (1.5) is sufficient for the discreteness of the spectrum. In the same situation V.P. Maslov [47] was able to provide a beautiful necessary and sufficient condition which generalizes the one-dimensional Molchanov theorem and is easy to formulate but not so easy to check for an operator-valued potential because it includes an operator valued energy-type estimate. R. Kleine [39] extended Maslov's result to the case when the semiboundedness condition on V is replaced by the Brinck condition (6.12). Finally J. Brüning [12] introduced a most general "coerciveness" assumption on V which is weaker than all the conditions above (in particular the weakest Zelenko condition in the scalar case) but still allows to establish a necessary and sufficient Maslov type condition of the discreteness of the spectrum. J. Brüning obtained applications of his abstract result to the Laplacian on manifolds with nice ends (with the conditions of the discreteness of the spectrum formulated e.g. in terms of Ricci curvature and mean curvature).

The first mathematical result about the discreteness of spectrum for the multidimensional Schrödinger operator (in $\mathbb{R}^n$) seems to be due to K. Friedrichs [30] who proved that the condition $V(x) \to +\infty$ as $x \to \infty$ is sufficient for $\sigma = \sigma_d$. (This proof, which is now considered elementary, can be also found e.g. in the books [74], [32], [64], [4].)

Molchanov's multidimensional result from [59] and further work by V.G. Maz'ya [52], [53], [55] revealed relevance of capacity in spectral theory. Further developments in the description of the spectrum under the Molchanov condition (in $\mathbb{R}^n$) are due to G.V. Rozenblum who established two-sided estimates for the eigenvalues distribution function and (under some mild additional restrictions) even the asymptotic for the Laplace and polyharmonic operators [66], as well as for the Schrödinger type operators [67]. The estimates themselves are formulated in terms of capacity, but simple (though strong) sufficient measure conditions follow.

Perturbation arguments allow to obtain some sufficient conditions of the discreteness of the spectrum in case when the potential V is allowed to be not semibounded below (see e.g. [64], Sect.XIII.14).

Other examples of usefulness of capacity in spectral theory were found by J. Rauch and M. Taylor [63] and G. Courtois [23]. In particular, G. Courtois established perturbation type estimates of eigenvalues of the Laplacian $-\Delta$ on $X \setminus A$ (here X is a closed compact Riemannian manifold, A is its compact subset) in terms of capacity of A, which is defined as follows:

$$\operatorname{cap}(A) = \inf \left\{ \int_X |\nabla u|^2 d\mu \,\middle|\, u \in C^\infty(X),\ \int_X u d\mu = 0,\ u = 1 \text{ near } A \right\}.$$

For example, he proved that

$$C^{-1} \operatorname{cap}(A) \le \lambda_1(X \setminus A) \le C \operatorname{cap}(A),$$

$$0 \le \lambda_k(X \setminus A) - \lambda_k \le C_k(\operatorname{cap}(A))^{1/2}.$$

(Here λ_k is the kth eigenvalue of $-\Delta$ on X, in particular $\lambda_1 = 0$, $\lambda_k(X \setminus A)$ is the kth Dirichlet eigenvalue of $X \setminus A$, multiplicities counted.) The convergence $\lambda_k(X \setminus A) \to \lambda_k$ as $\operatorname{cap}(A) \to 0$ (without estimate of remainder) follows already from the results of [63] where also other applications of capacity are given.

Some conditions of discreteness of the spectrum for the Laplacian on Riemannian manifolds were established in the papers in [3], [10], [11], [25], [26], [27].

In particular, A. Baider [3] considered some special manifolds which are warped products and used separation of variables which sometimes gives necessary and sufficient condition for the discreteness of the spectrum.

R. Brooks [10], [11] established lower and upper estimates for the bottom of the essential spectrum of the Laplacian in geometric terms, namely through the volume growth and Cheeger's isoperimetric constant. These estimates give in particular some necessary and some sufficient conditions for the spectrum to be discrete. For example, the spectrum is never discrete for complete Riemannian manifolds with infinite volume and with polynomial growth of the volumes of the balls. Some of the Brooks results were recently generalized by L. Notarantonio [60] to operators generated by regular Dirichlet forms.

H. Donnelly and P. Li [27] proved that the spectrum of the Laplacian on a complete simply connected negatively curved manifold M is discrete provided the sectional curvature tends to $-\infty$ at infinity. If $\dim M = 2$ then instead of the simply connectedness it is sufficient to require that the fundamental group $\pi_1(M)$ is finitely generated. The proof is based on a comparison result and isoperimetric inequalities technique by J. Cheeger [15] and S.T. Yau [80] (see also I. Chavel [14], Sect.IV.3). H. Donnelly in [25] established somewhat inverse results, proving existence of the essential spectrum of the Laplacian with some estimates of this spectrum. For example he proved that if M is non-compact, complete and has Ricci curvature bounded below by $-(n-1)c$, where $n = \dim M$, $c \ge 0$, then the essential spectrum intersects the interval $[0, (n-1)^2 c/4]$. It is interesting that no restriction on the injectivity radius is needed for this result. The proof relies on a comparison theorem by S.Y. Cheng [17] and some other similar comparison theorems.

The Rozenblum-Cwikel-Lieb bound (see e.g. [64], Sect.XIII.12) provides an estimate for the number of eigenvalues of a Schrödinger operator below some level E, whereas some essential spectrum above E can be present. In particular, it gives a condition that the spectrum below E is discrete. Recently D. Levin and M. Solomyak [43] proved that this bound holds in a very general context of Markov generators, in particular for the Schrödinger operators on manifolds where Ricci curvature is bounded below. The proof relies on the global Sobolev estimate (see e.g. [14], Sect.IV.3) which was proved in this context by L. Saloff-Coste [69], based on the ideas of P. Li and S.T. Yau [45]. Even more general result was obtained by G.V. Rozenblum and M.Z. Solomyak [68] in terms of the heat kernel behavior at 0 and ∞.

Note that the V.G.Maz'ya and M. Otelbaev result [57] (see also [55], Sect.12.5) which gives a two-sided estimate (in terms of capacity) for the bottom of the essential spectrum for the Schrödinger operator (and more general ones) in $\mathbb{R}^n$, implies in particular conditions for the spectrum below some level to be discrete.

Let us recall that discreteness of the spectrum for a self-adjoint operator is equivalent to the compactness of its resolvent. Hence a possibility to make the statement $\sigma = \sigma_d$ more precise is to establish that the resolvent belongs to a Schatten class $\mathcal{S}_p$ of compact operators, $1 \leq p < \infty$. For the Dirichlet spectrum of the Laplacian in a domain $G \subset \mathbb{R}^n$ this was done by E.B. Davies [24] in terms of convergence of some integrals, namely

$$\int_G [\mathrm{dist}(x, \partial G)]^\alpha dx < \infty,$$

where α depends on p. (Some related results for manifolds were obtained by M. Lianantonakis [46].)

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