

Discreteness of spectrum for the magnetic Schrödinger operators.

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Abstract

We consider a magnetic Schrödinger operator H in \mathbb{R}^n or on a Riemannian manifold M of bounded geometry. Sufficient conditions for the spectrum of H to be discrete are given in terms of behavior at infinity for some effective potentials V_{eff} which are expressed through electric and magnetic fields. These conditions can be formulated in the form $V_{eff}(x) \rightarrow +\infty$ as $x \rightarrow \infty$. They generalize the classical result by K.Friedrichs (1934), and include earlier results of J. Avron, I. Herbst and B. Simon (1978), A. Dufresnoy (1983) and A. Iwatsuka (1990) which were obtained in the absence of an electric field. More precise sufficient conditions can be formulated in terms of the Wiener capacity and extend earlier work by A.M. Molchanov (1953) and V. Kondrat'ev and M. Shubin (1999) who considered the case of the operator without a magnetic field. These conditions become necessary and sufficient in case there is no magnetic field and the electric potential is semi-bounded below.

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1 Introduction and main results

1.1 Notations and preliminaries

The main object of this paper is a magnetic Schrödinger operator in \mathbb{R}^n and its generalizations. In the simplest case it has the form

$$(1.1) \quad H_{a,V} = \sum_{j=1}^n P_j^2 + V,$$

where

$$(1.2) \quad P_j = \frac{1}{i} \frac{\partial}{\partial x^j} + a_j,$$

and $a_j = a_j(x)$, $V = V(x)$, $x = (x^1, \dots, x^n) \in \mathbb{R}^n$. We assume that a_j and V are real-valued functions.

For simplicity we will assume now that $a_j \in C^1(\mathbb{R}^n)$, $V \in L_{loc}^\infty(\mathbb{R}^n)$ (the later means that V is measurable and locally bounded). Then $H_{a,V}$ is well defined on $C_c^\infty(\mathbb{R}^n)$ (the set of all complex-valued C^∞ functions with a compact support in \mathbb{R}^n), and it is an unbounded symmetric operator in $L^2(\mathbb{R}^n)$.

We will always impose (explicitly or implicitly) conditions which insure that the operator $H_{a,V}$ is essentially self-adjoint. For instance, the condition $V \geq 0$ is sufficient (see e.g. H. Leinfelder and C.G. Simader [26] where this is proved under most general local regularity conditions on a_j and V). But some negative potentials (even mildly blowing up to $-\infty$ when $x \rightarrow \infty$) will do as well - see e.g. T. Ikebe and T. Kato [15], A. Iwatsuka [19], M. Shubin [36] for several versions of this fact. For the sake of convenience of the reader we give in Section 2 a very short proof of the fact which is important for us: the semi-boundedness below for the operator $H_{a,V}$ (on $C_c^\infty(\mathbb{R}^n)$) implies that it is essentially self-adjoint (this is an extension of the Povzner–Wienholtz theorem – see [32], [45], [12], [38]) and it is proved for the case of operators on any complete Riemannian manifold in [37]. We will also denote by $H_{a,V}$ the corresponding self-adjoint operator in $L^2(\mathbb{R}^n)$.

Actually the condition $V \in L_{loc}^\infty(\mathbb{R}^n)$ is not necessary for our study. For example, it will be sufficient to have $V \in L_{loc}^2(\mathbb{R}^n)$ and locally semi-bounded below. Moreover, it is sufficient to have $V \in L_{loc}^1(\mathbb{R}^n)$ and locally semi-bounded below. In this case we have to impose conditions which guarantee that the corresponding quadratic form $h_{a,V}$ is semi-bounded below and consider the operator defined by this form. This does not make any difference in our arguments because we can work with the quadratic form only. But we prefer not to get the reader distracted by unimportant details.

On the other hand working with capacities usually requires V to be locally semi-bounded below. So this condition often can not be removed.

We will say that a self-adjoint operator H in a Hilbert space \mathcal{H} has a *discrete spectrum* if its spectrum consists of isolated eigenvalues of finite multiplicities.

It follows that the only accumulation points of these eigenvalues can be $\pm\infty$. Equivalently we may say that H has a compact resolvent.

Our goal will be to provide conditions (mainly sufficient but sometimes necessary and sufficient) for the discreteness of the spectrum of $H_{a,V}$. We will abbreviate the discreteness of the spectrum of $H_{a,V}$ by writing $\sigma = \sigma_d$. Actually under the conditions we will impose the operator will be always semi-bounded below, so the only accumulation point of the eigenvalues will be $+\infty$.

Let us recall first some facts concerning the Schrödinger operator $H_{0,V} = -\Delta + V$ without magnetic field (i.e. the operator (1.1) with $a = 0$).

It is a classical result of K. Friedrichs [11] (see also e.g. [33], Theorem XIII.67, or [2], Theorem 3.1) that the condition

$$(1.3) \quad V(x) \rightarrow +\infty \quad \text{as } x \rightarrow \infty$$

implies $\sigma = \sigma_d$ (for $H_{0,V}$).

Now assume that

$$(1.4) \quad V(x) \geq -C$$

with a constant C , i.e. the potential V is semi-bounded below. Without loss of generality we can assume then that $V \geq 0$. Let us formulate a simple necessary condition for the discreteness of the spectrum. Denote by $B(x, r)$ the open ball with the radius $r > 0$ and the center at $x \in \mathbb{R}^n$. Then $\sigma = \sigma_d$ for $H_{0,V}$ implies that for every fixed $r > 0$

$$(1.5) \quad \int_{B(x,r)} V(y)dy \rightarrow +\infty \quad \text{as } x \rightarrow \infty.$$

This observation was made in a remarkable paper by A. Molchanov [31] who proved that in case $n = 1$ this condition is in fact necessary and sufficient (assuming the semi-boundedness of the potential V).

More importantly, A. Molchanov found a necessary and sufficient condition for $\sigma = \sigma_d$ to hold, again assuming (1.4). This condition is intermediate between (1.3) and (1.5). It is formulated in terms of the Wiener capacity which we will denote cap (see e.g. [9, 24, 29] for necessary properties of the capacity and more details). In case $n = 2$ the capacity of a set $F \subset B(x, r)$ is always taken relative to a ball $B(x, R)$ of a fixed radius $R > r$. (Expositions of Molchanov's and more general results can be found in [9, 24, 29].)

A. Molchanov proved that $H_{0,V}$ has a discrete spectrum if and only if there exist $c > 0$ and $r_0 > 0$ such that for any $r \in (0, r_0)$

$$(M_c) \quad \inf_F \left\{ \int_{B(x,r) \setminus F} V(y)dy \mid \text{cap}(F) \leq c \cdot \text{cap}(B(x, r)) \right\} \rightarrow +\infty \text{ as } x \rightarrow \infty.$$

(In this case we will say that the function V satisfies (M_c) , or that the Molchanov condition (M_c) holds for V . Later we will impose this condition on some other functions.) Note that (M_c) implies $(M_{c'})$ for any $c' < c$. Hence we can equivalently write that (M_c) is satisfied for all $c \in (0, c_0)$ with a positive c_0 . In fact

A. Molchanov provides a particular value of c (e.g. $c = 2^{-2n-6}$ would do – see [24]), though it is by no means precise.

Note also that $\text{cap}(B(x, r))$ can be explicitly calculated. It equals $c_n r^{n-2}$ if $n \geq 3$. If $n = 2$ then $\text{cap}(B(x, r))$ asymptotically equals $c_2 (\log(1/r))^{-1}$ as $r \rightarrow 0$. Hence in the formulation of the Molchanov condition (M_c) we can replace $\text{cap}(B(x, r))$ by r^{n-2} if $n \geq 3$ and by $(\log(1/r))^{-1}$ if $n = 2$.

A simple argument given in [1] (see also Sect.3 of this paper) shows that if $H_{0,V}$ has a discrete spectrum, then the same is true for $H_{a,V}$ whatever the magnetic potential a . Therefore the condition (M_c) together with (1.4) is sufficient for the discreteness of spectrum of $H_{a,V}$. This means that a magnetic field can only improve the situation from our point of view. Papers by J. Avron, I. Herbst and B. Simon [1], A. Dufresnoy [8] and A. Iwatsuka [18] provide some quantitative results which show that even in case $V = 0$ the magnetic field can make the spectrum discrete. In this paper we will improve the results of the above mentioned papers. In particular we will add the capacity into the picture, so in many cases our conditions become necessary and sufficient in case when there is no magnetic field, i.e. when $a = 0$. We will also make both electric and magnetic fields work together to achieve the discreteness of spectrum.

Unfortunately we can not provide efficient necessary and sufficient conditions of the discreteness of the spectrum when both fields are present (or even if the magnetic field only is present). The conditions which we can give always contain some hypotheses which are hard to check (unless $a = 0$ when they become trivial). Some of these conditions will be discussed in a future continuation of this paper.

It is convenient to consider the magnetic potential as a 1-form a with components a_j :

$$(1.6) \quad a = a_j dx^j,$$

where we use the Einstein summation convention (i.e. the summation over all repeated suffices is understood). Now the *magnetic field* is a 2-form B which is defined as

$$(1.7) \quad B = da = \frac{\partial a_j}{\partial x^k} dx^k \wedge dx^j = \frac{1}{2} B_{jk} dx^j \wedge dx^k,$$

where $B_{kj} = -B_{jk}$. Obviously

$$(1.8) \quad B = \sum_{j < k} B_{jk} dx^j \wedge dx^k,$$

and

$$(1.9) \quad B_{jk} = \frac{\partial a_k}{\partial x^j} - \frac{\partial a_j}{\partial x^k},$$

so in the standard vector analysis notation $B = \text{curl } a$. The functions B_{jk} will be called *components* of the magnetic field B .

In case $n = 2$ the magnetic field has essentially one non-trivial component $B_{12} = -B_{21}$ and in this case we will denote $B = B_{12}$.

We will need a norm of B which is defined as

$$(1.10) \quad |B| = \left(\sum_{j < k} |B_{jk}|^2 \right)^{1/2}.$$

Note that the components of the magnetic field show up in the commutation relations

$$(1.11) \quad [P_j, P_k] = \frac{1}{i} B_{jk},$$

where $[A, B] = AB - BA$ for operators A, B in the same Hilbert space (at the moment we assume that all operations are performed on the same domain, e.g. $C_c^\infty(\mathbb{R}^n)$ for P_j and P_k). The relation (1.11) allows to apply uncertainty principle type arguments in investigating the spectrum.

An important fact is the *gauge invariance* of the spectrum of $H_{a,V}$: this spectrum does not depend of the choice of the magnetic potential a provided the magnetic field B is fixed. Namely, if a, a' are two magnetic potentials with $da = da' = B$, then $\sigma(H_{a,V}) = \sigma(H_{a',V})$ for any V . To see this note that by the Poincaré Lemma (see e.g. 4.18 in [44]) we have $a' = a + d\phi$, where $\phi \in C^1(\mathbb{R}^n)$ is defined up to an additive constant and can be assumed real-valued. Then the corresponding operators

$$P'_j = \frac{1}{i} \frac{\partial}{\partial x^j} + a'_j$$

are related with P_j by the formulae

$$P'_j = e^{-i\phi} P_j e^{i\phi}.$$

Therefore

$$H_{a',V} = e^{-i\phi} H_{a,V} e^{i\phi},$$

and the operators $H_{a',V}$ and $H_{a,V}$ are unitarily equivalent, hence have the same spectra.

For the weakest requirements on the magnetic potential a the gauge invariance was established by H. Leinfelder [25].

By this reason it is more natural for spectral theory to formulate the conditions on $H_{a,V}$ in terms of B, V rather than a, V .

Let us assume that we are given a magnetic potential $a = a_j dx^j$, $a_j \in C^1(\mathbb{R}^n)$. For a function $u \in C^1(\mathbb{R}^n)$ (or, more generally, for a locally Lipschitz function) define *magnetic differential* as

$$(1.12) \quad d_a u = du + iua \in \Lambda^1(\mathbb{R}^n).$$

It is also convenient to identify this complex-valued 1-form with the corresponding complex vector field which is called *magnetic gradient*:

$$(1.13) \quad \nabla_a u = \left(\frac{\partial u}{\partial x^1} + ia_1 u, \dots, \frac{\partial u}{\partial x^n} + ia_n u \right) = (iP_1 u, \dots, iP_n u).$$

We will denote by $|\cdot|$ the usual euclidean norm of vectors or 1-forms.

1.2 Localization

Necessary and sufficient conditions of discreteness of spectrum for $H_{a,V}$ can be formulated in terms of bottoms of Dirichlet or Neumann spectra on balls of a fixed radius or on cubes of a fixed size. We will call these facts *localization* results. The first result of this kind about usual Schrödinger operators (without magnetic field) is due to A. Molchanov [31] (see also [24] for a more general theorem on manifolds). A. Iwatsuka [18] proved a localization theorem for magnetic Schrödinger operators.

The bottoms of Dirichlet and Neumann spectra for the operator $H_{a,V}$ in an open set $\Omega \subset \mathbb{R}^n$ are defined in terms of its quadratic form which we will denote $h_{a,V}$:

$$(1.14) \quad h_{a,V}(u, u) = \int_{\Omega} (|\nabla_a u|^2 + V|u|^2) dx.$$

This form is well defined e.g. for all $u \in L^2(\Omega)$ such that $P_j u \in L^2(\Omega)$, $j = 1, \dots, n$, the derivatives are understood in the sense of distributions, and $V|u|^2 \in L^1(\Omega)$. In particular $h_{a,V}(u, u)$ is well defined for all $u \in C_c^\infty(\Omega)$. Denote by (u, v) the usual scalar product of u and v in $L^2(\Omega)$.

It is easy to check that the following gauge invariance relation holds:

$$(1.15) \quad h_{a+d\phi, V}(u, u) = h_{a, V}(e^{i\phi} u, e^{i\phi} u),$$

for any $\phi \in C^1(\mathbb{R}^n)$ and u as above.

Now we can define

$$(1.16) \quad \lambda(\Omega; H_{a,V}) = \inf_u \left\{ \frac{h_{a,V}(u, u)}{(u, u)}, u \in C_c^\infty(\Omega) \setminus 0 \right\},$$

$$(1.17) \quad \mu(\Omega; H_{a,V}) = \inf_u \left\{ \frac{h_{a,V}(u, u)}{(u, u)}, u \in (C^\infty(\Omega) \setminus \{0\}) \cap L^2(\Omega), \int_{\Omega} V|u|^2 dx > -\infty \right\},$$

i.e. $\lambda(\Omega; H_{a,V})$ and $\mu(\Omega; H_{a,V})$ are bottoms of the Dirichlet and Neumann spectra (of $H_{a,V}$) respectively, in the usual variational understanding (see e.g. [6], [22]).

The relation (1.15) obviously implies that the numbers $\lambda(\Omega; H_{a,V})$ and $\mu(\Omega; H_{a,V})$ are gauge invariant, i.e. they do not change if we replace a by $a + d\phi$ for any $\phi \in C^1(\mathbb{R}^n)$.

The following theorem slightly extends the result of A. Iwatsuka [18] removing the requirement $V \geq 0$ and allowing non-continuous minorant functions.

Theorem 1.1 *The following conditions are equivalent:*

- (a) $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum;
- (b) $\lambda(B(x, r); H_{a,V}) \rightarrow +\infty$ as $x \rightarrow \infty$ for any fixed $r > 0$;
- (c) there exists $r > 0$ such that $\lambda(B(x, r); H_{a,V}) \rightarrow +\infty$ as $x \rightarrow \infty$;
- (d) there exists a real valued function $\Lambda \in C(\mathbb{R}^n)$ such that $\Lambda(x) \rightarrow +\infty$ as $x \rightarrow \infty$ and the operator inequality

$$(1.18) \quad H_{a,V} \geq \Lambda(x)$$

holds in the sense of quadratic forms (on $C_c^\infty(\mathbb{R}^n)$);

- (e) there exists a measurable function $\Lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\Lambda(x) \rightarrow +\infty$ as $x \rightarrow \infty$ and (1.18) holds.

If we additionally assume that the electric potential V is semi-bounded below then we can add the bottoms of the Neumann spectrum to the picture as was first done by A. Molchanov [31], though without magnetic field. This will be important for some arguments in this paper.

Theorem 1.2 *If V is semi-bounded below then the conditions in Theorem 1.1 are also equivalent to the following conditions:*

- (f) $\mu(B(x, r); H_{a,V}) \rightarrow +\infty$ as $x \rightarrow \infty$ for any fixed $r > 0$;
- (g) there exists $r > 0$ such that $\mu(B(x, r); H_{a,V}) \rightarrow +\infty$ as $x \rightarrow \infty$.

Finally we can weaken the requirement on Λ by use of capacity through the Molchanov condition:

Theorem 1.3 *Let us assume that V is semi-bounded below. Then the conditions (a) – (g) in Theorems 1.1 and 1.2 are equivalent to the existence of a measurable function $\Lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

- 1) Λ is semi-bounded below and satisfies (M_c) with some $c > 0$;
- 2) the operator inequality $H_{a,V} \geq \Lambda(x)$ holds in the same sense as in Theorem 1.1.

For $\Lambda = V$ and $a = 0$ this gives a sufficiency part of the Molchanov theorem. It also implies the following result from [1]:

Corollary 1.4 *If V is semi-bounded below and $H_{0,V}$ has a discrete spectrum (or, equivalently, V satisfies (M_c) with some $c > 0$), then $H_{a,V}$ also has a discrete spectrum for any magnetic potential a .*

Our proof of this statement is in fact purely variational and it is completely different from the proof given in [1] where semigroup methods are used. However the regularity requirements for a and V are much weaker in [1].

Finally let us formulate a convenient sufficient condition which does *not* require V or Λ to be semi-bounded below.

Theorem 1.5 *Assume that there exists a measurable function $\Lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the following conditions are satisfied:*

- 1) *the operator inequality*

$$(1.19) \quad H_{a,0} \geq \Lambda(x)$$

holds in the same sense as above;

- 2) *there exists $\delta \in [0, 1)$ such that the effective potential*

$$(1.20) \quad V_{\text{eff}}^{(\delta)}(x) = V(x) + \delta\Lambda(x)$$

is semi-bounded below and satisfies (M_c) with some $c > 0$.

Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

1.3 Sufficient conditions ($n = 2$)

The case $n = 2$ is much simpler than the general case because the magnetic field does not change direction (and may only change sign). We will identify the magnetic field with its component B_{12} . Note that by changing enumeration of coordinates we can change sign of $B = B_{12}$. Some uncertainty principle related arguments lead to the following fact established by J. Avron, I. Herbst and B. Simon [1]:

Theorem 1.6 *If $n = 2$ and $|B(x)| \rightarrow \infty$ as $x \rightarrow \infty$, then $\sigma = \sigma_d$ for $H_{a,0}$ (hence for $H_{a,V}$ with arbitrary $V \geq 0$).*

This is the simplest result which shows that magnetic field alone may cause a localization (i.e. a discrete spectrum) for a quantum particle. Classically this can be understood from the fact that a strong magnetic field causes a fast rotation of a charged moving particle without changing its kinetic energy, and in this way the field impedes possible escape of the particle to infinity.

Note that $|B(x)| \rightarrow \infty$ means in fact that either B or $-B$ tend to $+\infty$ as $x \rightarrow \infty$. Hence the following theorem improves the result above:

Theorem 1.7 *Assume that the following conditions are satisfied:*

- (a) *there exists $C > 0$ such that $B(x) \geq -C$, $x \in \mathbb{R}^2$;*
(b) *the Molchanov condition (M_c) is satisfied for $B(x)$ or, equivalently, for $|B(x)|$ (instead of $V(x)$) with some $c > 0$.*

Then $H_{a,0}$ has a discrete spectrum.

The simplest explicit result which takes into account both electric and magnetic field is given by the following

Theorem 1.8 *Assume that $n = 2$ and there exists $\delta \in [-1, 1]$ such that for the effective potential*

$$(1.21) \quad V_{\text{eff}}^{(\delta)}(x) = V(x) + \delta B(x)$$

we have $V_{\text{eff}}^{(\delta)}(x) \rightarrow +\infty$ as $x \rightarrow \infty$. Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

V. Ivrii noticed that the range of possible δ is precise here: the conclusion does not hold if we take any $\delta \notin [-1, 1]$.

The following theorem strengthens Theorem 1.7 by taking into account the influence of the electric potential V :

Theorem 1.9 *Assume that $n = 2$ and there exists $\delta \in (-1, 1)$ such that the effective potential V_{eff} given by (1.21) is semi-bounded below and satisfies the Molchanov condition (M_c) with some $c > 0$. Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.*

Clearly Theorem 1.9 implies Theorem 1.7 (take $V = 0$). It also implies the sufficiency of the Molchanov condition in case $B = 0$. Note however that the conditions of the theorem do not imply any growth of V or B . If $|B|$ itself tends to ∞ as $x \rightarrow \infty$, then V is even allowed to go to $-\infty$, though in this case $|V|$ must be dominated by $\delta|B|$ with some positive $\delta < 1$.

Theorem 1.9 also implies Theorem 1.8 except for the extreme values $\delta = \pm 1$.

Unfortunately we are not aware whether any of the conditions in Theorem 1.9 is necessary (though they are necessary if $B = 0$ and V is semi-bounded below due to the Molchanov result quoted above).

Even Theorem 1.8, which does not require any use of capacity and can be proved by elementary means (see Sect.5.1), seems to be absent in the literature. A stronger result without capacity can be obtained if we replace capacity by the Lebesgue measure. Namely, let us formulate the corresponding condition for a semi-bounded below function V : there exists $r_0 > 0$ such that for any $r \in (0, r_0)$

$$(\tilde{M}_{c,N}) \quad \inf_F \left\{ \int_{B(x,r) \setminus F} V(y) dy \mid \text{mes}(F) \leq cr^N \right\} \rightarrow +\infty \text{ as } x \rightarrow \infty.$$

It follows from a well-known estimate of measure by capacity, that for any $c > 0$ and $N > 0$ the condition $(\tilde{M}_{c,N})$ implies $(M_{c'})$ for some $c' > 0$ (see the proof of part 2) of Theorem 6.1 in [24]). Therefore Theorem 1.9 implies the following

Corollary 1.10 *Assume that there exists $\delta \in (-1, 1)$ such that the effective potential $V_{\text{eff}}^{(\delta)}$ given by (1.21) is semi-bounded below and satisfies $(\tilde{M}_{c,N})$ with some $c > 0$ and $N > 0$. Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.*

Finally note that an elementary argument given in the proof of Corollary 6.2 in [24] gives a sufficient condition which is stronger than $(\tilde{M}_{c,N})$ but very easy to check.

Corollary 1.11 *Assume that there exists $\delta \in (-1, 1)$ such that $V_{eff}^{(\delta)}$ is semi-bounded below and for any $A > 0$ and any small $r > 0$*

$$(1.22) \quad \text{mes} \{y \mid y \in B(x, r), V_{eff}^{(\delta)}(y) \leq A\} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

1.4 Sufficient conditions ($n \geq 3$)

The behavior of the spectrum of the magnetic Schrödinger operator in dimensions $n \geq 3$ is much more complicated than in dimension 2 because of possible varying direction of B . In particular none of the results formulated above for $n = 2$ holds for $n \geq 3$. A. Dufresnoy [8] gave the first example of the operator $H_{a,0}$ with

$$(1.23) \quad |B(x)| \rightarrow \infty \quad \text{as } x \rightarrow \infty,$$

and yet with non-compact resolvent (or, equivalently, with non-discrete spectrum) in an arbitrary dimension $n \geq 3$. J. Avron, I. Herbst and B. Simon [1] gave sufficient conditions for the discreteness of the spectrum of $H_{a,0}$ which in addition to (1.23) require that the direction of B varies sufficiently slowly. A more explicit condition of this kind (in terms of estimates for derivatives of the direction of B) was given in [8]. Later A. Iwatsuka [18] improved these results, giving almost precise estimates of this kind. He also produced a series of spectacular examples. One of them shows that no growth condition for $|B(x)|$ (i.e. a condition of the form $|B(x)| \geq \rho(x)$ with a fixed continuous function ρ) would suffice for the discreteness of the spectrum of $H_{a,0}$. This is in drastic contrast with the results for $n = 2$ formulated above. Another example given in [18] shows that the condition (1.23) is also not necessary for the discreteness of spectrum of $H_{a,0}$. (This is of course less surprising because you can expect that integrally small perturbations of B should not generally affect the discreteness of spectrum.)

However we will show in Sect.5 that some explicit sufficient conditions for the discreteness of spectrum can still be formulated in terms of effective potentials similarly to the case $n = 2$. The appropriate effective potentials will include both electric and magnetic fields. They will incorporate information about the direction of the magnetic field. The above mentioned results of J. Avron, I. Herbst and B. Simon, A. Dufresnoy and A. Iwatsuka will follow if we impose additional conditions on the direction of the magnetic field. These conditions allow us to simplify the form of the effective potential.

Here we will formulate just one result coming from this approach (others can be found in Sect.5).

By $\text{Lip}(X)$ we will denote the set of all Lipschitz functions on any metric space. We will only use X which are locally compact. In this case $\text{Lip}_{loc}(X)$

will denote the space of functions which are locally Lipschitz on X . Recall that Lipschitz functions on an open subset in \mathbb{R}^n are exactly the ones which have bounded distributional derivatives (see e.g. [29]).

Let us define a smoothed direction of the magnetic field as follows:

$$(1.24) \quad A_{jk}(x) = \chi(|B(x)|) \frac{B_{jk}(x)}{|B(x)|},$$

where $\chi \in \text{Lip}([0, \infty))$, $\chi(r) = 0$ if $r \leq 1/2$, $\chi(r) = 1$ if $r \geq 1$, $\chi(r) = 2r - 1$ if $1/2 \leq r \leq 1$, so $0 \leq \chi(r) \leq 1$ and $|\chi'(r)| \leq 2$ for all r .

Theorem 1.12 *Let us assume that $B_{jk} \in \text{Lip}(\mathbb{R}^n)$ for all j, k ; A_{jk} are defined by (1.24), and a positive measurable function $X(x)$ in \mathbb{R}^n satisfies*

$$(1.25) \quad \sum_{k=1}^n \left| \frac{\partial A_{kj}(x)}{\partial x^k} \right| \leq X(x), \quad x \in \mathbb{R}^n,$$

for all j . Then for any $\varepsilon > 0$ and $\delta \in [0, 1)$ define an effective potential

$$(1.26) \quad V_{\text{eff}}^{(\delta, \varepsilon)}(x) = V(x) + \frac{\delta}{n-1+\varepsilon} |B(x)| - \frac{n\delta}{4\varepsilon(n-1+\varepsilon)} X^2(x).$$

If there exist $\varepsilon > 0$ and $\delta \in [0, 1)$, such that $V_{\text{eff}}^{(\delta, \varepsilon)}$ is bounded below and satisfies the Molchanov condition (M_c) with some $c > 0$ (in particular, this holds if $V_{\text{eff}}^{(\delta, \varepsilon)}(x) \rightarrow +\infty$ as $x \rightarrow \infty$), then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Note that in case $B = 0$ and V semi-bounded below the condition on V in this theorem becomes necessary and sufficient due to the Molchanov theorem.

In Section 5 we will establish that imposing different regularity conditions on B , e.g. as in [1, 8, 18], leads to different types of effective potentials so that results similar to Theorem 1.9 hold.

Here we will give the simplest example of this kind. As explained above it is natural to impose some a priori conditions of regularity on B . We will mainly formulate them in the form of estimates for derivatives of B . We will always assume that $B \in \text{Lip}_{loc}$, i.e. $B_{jk} \in \text{Lip}_{loc}$ for all j, k . Following A. Iwatsuka [18] we will use the estimates of the form

$$(B_\alpha) \quad |\nabla B(x)| \leq C(1 + |B(x)|)^\alpha, \quad x \in \mathbb{R}^n,$$

where $\alpha > 0$, $C > 0$ and ∇B means vector whose components are all possible first order derivatives $\partial B_{jk} / \partial x^l$. We will write that B satisfies (B_α) if there exists $C > 0$ such that this estimate is satisfied with the given α . A little bit stronger condition used in [8] and [18] is

$$(B_\alpha^0) \quad |\nabla B(x)|(1 + |B(x)|)^{-\alpha} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

A. Dufresnoy [8] proved that the conditions (1.23) and $(B_{3/2}^0)$ imply that the spectrum of $H_{a,0}$ is discrete. In fact he proved that instead of $(B_{3/2}^0)$ it is sufficient to require a slightly weaker condition

$$\left| \nabla \left(\frac{B(x)}{|B(x)|} \right) \right| \cdot |B(x)|^{-1/2} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

A. Iwatsuka [18] proved that in fact (B_2^0) together with (1.23) imply the discreteness of spectrum for $H_{a,0}$. He also provided an example which shows that (B_2^0) can not be replaced by (B_2) (hence it can not be replaced by (B_α) or (B_α^0) with any $\alpha > 2$).

The following theorem compared with the above mentioned results takes into account the behavior of V .

Theorem 1.13 *Assume that*

(a) B satisfies $(B_{3/2}^0)$,

and

(b) there exists $\delta \in [0, 1)$ such that the effective potential

$$(1.27) \quad V_{\text{eff}}^{(\delta)} = V + \frac{\delta}{n-1} |B|$$

is semi-bounded below and satisfies the Molchanov condition (M_c) with a small $c > 0$ (possibly depending on δ).

Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Corollary 1.14 *Assume that*

(a) B satisfies $(B_{3/2}^0)$

and

(b) there exists $\delta \in [0, 1)$ such that $V_{\text{eff}}^{(\delta)}(x) \rightarrow +\infty$ as $x \rightarrow \infty$ for the effective potential $V_{\text{eff}}^{(\delta)}$ defined by (1.27).

Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Due to the arguments given in Sect.6.1 of [24] it is also possible to replace capacity by the Lebesgue measure. Namely, let us introduce the corresponding condition for a semi-bounded below function V :

$$(\tilde{M}_c) \quad \inf_F \left\{ \int_{B(x,r) \setminus F} V(y) dy \mid \text{mes}(F) \leq cr^n \right\} \rightarrow +\infty \text{ as } x \rightarrow \infty.$$

As shown in Sect.6.1 of [24], for any $c > 0$ there exists $c' > 0$ such that (\tilde{M}_c) implies $(M_{c'})$. Therefore we have the following

Corollary 1.15 *Assume that B satisfies $(B_{3/2}^0)$ and*

(b) there exists $\delta \in [0, 1)$ such that the effective potential (1.27) is semi-bounded below and there exists $c > 0$ such that the condition (\tilde{M}_c) holds for $V_{\text{eff}}^{(\delta)}$ (instead of V);

Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Finally let us formulate an analogue of Corollary 1.11 for $n \geq 3$.

Corollary 1.16 *Assume that B satisfies $(B_{3/2}^0)$ and*

(b) there exists $\delta \in [0, 1)$ such that the effective potential (1.27) is semi-bounded below and for any $A > 0$ and any small $r > 0$

$$(1.28) \quad \text{mes} \{y \mid y \in B(x, r), V_{\text{eff}}^{(\delta)}(y) \leq A\} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Note that if we assume that V is semi-bounded below, then we can always take $V + |B|$ as an effective potential in all statements above.

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2 Essential self-adjointness

The goal of this section is to give a simple proof of the following simplest version of the theorem on essential self-adjointness of any semi-bounded magnetic Schrödinger operator (see [32, 45, 12, 38, 37] for other versions).

Theorem 2.1 *Assume that $a_j \in \text{Lip}_{loc}(\mathbb{R}^n)$, $V \in L_{loc}^\infty(\mathbb{R}^n)$ and the corresponding magnetic Schrödinger operator $H_{a,V}$ is semi-bounded below on $C_c^\infty(\mathbb{R}^n)$, i.e. there exists a constant $C \in \mathbb{R}$ such that*

$$(2.1) \quad (H_{a,V}u, u) \geq -C(u, u), \quad u \in C_c^\infty(\mathbb{R}^n).$$

Then $H_{a,V}$ is essentially self-adjoint.

Proof. We will extend the Wienholtz proof of the Povzner theorem as it is explained by I.M. Glazman [12].

Let us recall that if $g \in \text{Lip}(\Omega)$ where Ω is an open subset in \mathbb{R}^n , then $\partial g / \partial x^j \in L^\infty(\Omega)$, $j = 1, \dots, n$ (see e.g. [29], Sect.1.1), where the derivatives are understood in the sense of distributions (but also exist almost everywhere). This implies that the operator $H_{a,V}$ is well defined on $C_c^\infty(\mathbb{R}^n)$ (and maps this space into $L^2(\mathbb{R}^n)$) as well as on $L^2(\mathbb{R}^n)$ (which it maps to the space of distributions on \mathbb{R}^n).

Note that adding $(C+1)I$ to $H_{a,V}$ we can assume that $H_{a,V} \geq I$ on $C_c^\infty(\mathbb{R}^n)$, i.e.

$$(2.2) \quad h_{a,V}(u, u) \geq (u, u), \quad u \in C_c^\infty(\mathbb{R}^n).$$

If this is true then it is well known (see e.g. [12]) that the essential self-adjointness of $H_{a,V}$ is equivalent to the fact that the equation

$$(2.3) \quad H_{a,V}u = 0$$

has no non-trivial solutions in $L^2(\mathbb{R}^n)$ (understood in the sense of distributions).

Assume that u is such a solution. First note that it is in $W_{loc}^{2,2}(\mathbb{R}^n)$ (i.e. has distributional derivatives of order ≤ 2 in $L_{loc}^2(\mathbb{R}^n)$) due to a simple elliptic regularity argument (see e.g. Lemma 4.1 in [36] for more details).

Let us take a cut-off function $\phi_R \in C_c^\infty(\mathbb{R}^n)$ with the following properties:

$$\begin{aligned} 0 &\leq \phi_R \leq 1; \\ \phi_R &= 1 \text{ on } B(0, R) \text{ and } 0 \text{ on } \mathbb{R}^n \setminus B(0, 2R); \\ \varepsilon_R &:= \sup_{x \in \mathbb{R}^n} |\nabla \phi_R(x)| \rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned}$$

Then denoting $u_R = \phi_R u$ we see that u_R is in the domain of the minimal operator associated with $H_{a,V}$, hence

$$(2.4) \quad \|u_R\|^2 \leq (H_{a,V}u_R, u_R).$$

Let us calculate $H_{a,V}u_R$ using the Leibniz type formula for P_j :

$$(2.5) \quad P_j(fg) = (P_j f)g + f(D_j g),$$

where $D_j = -i\partial/\partial x^j$. Applying this formula twice to calculate $P_j^2(\phi_R u)$ and summing up we easily obtain due to (2.3):

$$(2.6) \quad \begin{aligned} H_{a,V}u_R &= \phi_R H_{a,V}u - 2(\nabla \phi_R) \cdot (\nabla_a u) - u \Delta \phi_R \\ &= -2(\nabla \phi_R) \cdot (\nabla_a u) - u \Delta \phi_R \\ &= -2\nabla \phi_R \cdot \nabla u - 2i(a \cdot \nabla \phi_R)u - u \Delta \phi_R. \end{aligned}$$

Therefore due to 2.4 we have

$$(2.7) \quad \|\phi_R u\|^2 \leq \int_{\mathbb{R}^n} (-2\nabla \phi_R \cdot \nabla u - 2i(a \cdot \nabla \phi_R)u - u \Delta \phi_R) \phi_R \bar{u} dx.$$

Since $(H_{a,V}u_R, u_R)$ is real, we can replace the right hand side here by the complex conjugate expression. Adding the two estimates obtained in such a way and dividing by 2 we see that the term with the magnetic potential a cancels and we get, applying integration by parts:

$$\begin{aligned}
\|\phi_R u\|^2 &\leq \int_{\mathbb{R}^n} [-\phi_R(\nabla\phi_R) \cdot (\bar{u}\nabla u + u\nabla\bar{u}) - \phi_R(\Delta\phi_R)|u|^2] dx \\
&= \int_{\mathbb{R}^n} [-\phi_R(\nabla\phi_R) \cdot \nabla(|u|^2) - \phi_R(\Delta\phi_R)|u|^2] dx \\
&= \int_{\mathbb{R}^n} [\phi_R(\Delta\phi_R)|u|^2 + |\nabla\phi_R|^2|u|^2 - \phi_R(\Delta\phi_R)|u|^2] dx \\
&= \int_{\mathbb{R}^n} |\nabla\phi_R|^2|u|^2 dx.
\end{aligned}$$

In particular we obtain using the conditions on ϕ_R above:

$$\int_{B(0,R)} |u|^2 dx \leq \varepsilon_R^2 \int_{B(0,2R)} |u|^2 dx.$$

Allowing R to go to $+\infty$ we see that $\|u\|^2 = 0$, hence $u \equiv 0$. \square

Remark. The local condition $V \in L_{loc}^\infty(\mathbb{R}^n)$ can be considerably weakened. For example, it is sufficient to require that $V = V_+ + V_-$, where $V_+ \geq 0$, $V_+ \in L_{loc}^2(\mathbb{R}^n)$, $V_- \leq 0$, $V_- \in L_{loc}^p(\mathbb{R}^n)$ with $p = 2$ if $n \leq 3$, $p > 2$ if $n = 4$, and $p = n/2$ if $n \geq 5$. (See e.g. [38, 37].)

3 Localization

In this section we will prove different localization theorems which were formulated in Sect.1.2 and provide an important preliminary material related to compactness arguments and estimates of the bottoms of the Dirichlet and Neumann spectra.

We will use notations from previous sections. In particular, a, V will always denote magnetic and electric potential with the same regularity as in Section 1.

We will start with the following elementary and well known diamagnetic inequality (see e.g. [23, 39, 28]):

Lemma 3.1 *Let a be an arbitrary magnetic potential (with components from C^1). Let u be a complex valued Lipschitz function in an open set $U \subset \mathbb{R}^n$. Then $|u|$ is also Lipschitz and*

$$(3.1) \quad |\nabla|u|| \leq |\nabla_a u| \quad a.e.,$$

where *a.e.* means almost everywhere with respect to the Lebesgue measure.

Let us assume that $H_{a,V}$ is bounded below, hence essentially self-adjoint due to Theorem 2.1. Without loss of generality we can assume hereafter that $H_{a,V} \geq I$ (or that the estimate (2.2) is satisfied).

The essential self-adjointness of $H_{a,V}$ means that $C_c^\infty(\mathbb{R}^n)$ is its *core*, i.e. the closure of $H_{a,V}$ from the initial domain $C_c^\infty(\mathbb{R}^n)$ is a self-adjoint operator in $L^2(\mathbb{R}^n)$. It follows that $C_c^\infty(\mathbb{R}^n)$ is also a core for the corresponding quadratic form.

Denote

$$(3.2) \quad \mathcal{L} = \{u \in C_c^\infty(\mathbb{R}^n) \mid h_{a,V}(u, u) \leq 1\}.$$

Lemma 3.2 $\sigma = \sigma_d$ if and only if \mathcal{L} is precompact in $L^2(\mathbb{R}^n)$.

Proof. The proof is the same as the proof of Lemma 2.2 in [24] and it is essentially abstract. Clearly $\sigma = \sigma_d$ is equivalent to saying that for $H = H_{a,V}$ we have

$$(3.3) \quad \{u \mid u \in \text{Dom}(H^{1/2}), \|H^{1/2}u\| \leq 1\}$$

is precompact in $L^2(\mathbb{R}^n)$. Here $\text{Dom}(H^{1/2})$ also coincide with the domain of the quadratic form which is the closure of $h_{a,V}$. Since $C_c^\infty(\mathbb{R}^n)$ is a core for the quadratic form too, we see that precompactness of the set (3.3) is equivalent to the precompactness of \mathcal{L} . \square

Lemma 3.3 (SMALL TAILS LEMMA) *Let us assume as above that $H_{a,V}$ is essentially self-adjoint and semi-bounded below so that (2.2) holds. Then $\sigma = \sigma_d$ if and only if the following small tails condition is satisfied: for any $\varepsilon > 0$ there exists $R > 0$ such that*

$$(3.4) \quad \int_{\mathbb{R}^n \setminus B(0,R)} |u|^2 dx < \varepsilon \text{ for any } u \in \mathcal{L},$$

or, in other words,

$$\int_{\mathbb{R}^n \setminus B(0,R)} |u|^2 dx \rightarrow 0 \text{ as } R \rightarrow \infty, \text{ uniformly in } u \in \mathcal{L}.$$

Proof. Again the proof is similar to the proof of Lemma 2.3 in [24], though a small additional argument is needed to avoid using semi-boundedness of V which was used in [24].

Clearly $\sigma = \sigma_d$ (or precompactness of \mathcal{L}) implies the small tails condition because any precompact set has a ε -net for any $\varepsilon > 0$.

Vice versa, assume that the small tails condition is fulfilled. Then the precompactness of \mathcal{L} would be equivalent to the precompactness of any restriction

$$\mathcal{L}_R = \{u|_{B(0,R)} \mid u \in \mathcal{L}\},$$

where $R > 0$. Note that the condition (2.2) implies that the set \mathcal{L}_R is bounded in $L^2(B(0,R))$. But then by the Sobolev–Kondrashov compactness theorem it is sufficient to establish a uniform $L^2(B(0,R))$ -boundedness of the gradients of functions $u \in \mathcal{L}_R$ (for any fixed $R > 0$). Since we assume that $a_j \in L_{loc}^\infty$, it is

sufficient to establish that the magnetic gradients $\nabla_a u$ are uniformly bounded in $L^2(B(0, R))$. This in turn follows from the definition of \mathcal{L} and uniform L^2 -boundedness of the functions $u \in \mathcal{L}$ combined with the local boundedness of the potential V . \square

Remark. The requirement $u \in C_c^\infty(\mathbb{R}^n)$ in the definition of \mathcal{L} (and in Lemmas 3.2 and 3.3) can be replaced by the requirement $u \in \text{Lip}_c(\mathbb{R}^n)$ (the set of all Lipschitz functions with compact support in \mathbb{R}^n) because the space $\text{Lip}_c(\mathbb{R}^n)$ is intermediate between $C_c^\infty(\mathbb{R}^n)$ and $\text{Dom}(H_{a,V}^{1/2})$.

Now let us take a covering of \mathbb{R}^n by balls $B(x_k, r)$, $k = 1, 2, \dots$, of fixed radius $r > 0$, such that this covering has a finite multiplicity. Now take a partition of unity on \mathbb{R}^n consisting of functions $e_k \in C_c^\infty(\mathbb{R}^n)$, $k = 1, 2, \dots$, such that $0 \leq e_k \leq 1$, $\text{supp } e_k \subset B(x_k, r)$, and

$$\sum_{k=1}^{\infty} e_k^2 = 1,$$

$$|\nabla e_k| \leq C, \quad k = 1, 2, \dots,$$

where C does not depend on k .

The main tool in proving the localization theorem is the following IMS localization formula

$$(3.5) \quad H_{a,V} = \sum_{k=1}^{\infty} J_k H_{a,V} J_k - \sum_{k=1}^{\infty} |\nabla e_k|^2,$$

where J_k is the multiplication operator by e_k in $L^2(\mathbb{R}^n)$. Proofs of different versions of this formula can be found in [42], [7] (Sect. 3.1), [35]. Formally only the case $a = 0$ is treated in [42, 7], though the proof works with arbitrary a . Much more general case (second order differential operators on manifolds) is considered in [35], Sect.3.

Now we will fix an operator $H_{a,V}$ and denote for brevity $\lambda(B(x, r)) = \lambda(B(x, r); H_{a,V})$, $\mu(B(x, r)) = \mu(B(x, r); H_{a,V})$.

Proof of Theorem 1.1. (a) \implies (b). Let us assume that (a) is satisfied and fix an arbitrary $r > 0$. According to Lemma 3.3 for any $\varepsilon > 0$ there exists $R > 0$ such that $|x| > R$ implies that

$$\int_{B(x,r)} |u(x)|^2 dx < \varepsilon,$$

as soon as $u \in C_c^\infty(B(x, r))$ and $h_{a,V}(u, u) \leq 1$. It follows that $\lambda(B(x, r)) \geq 1/\varepsilon$, which implies (b).

Clearly (b) \implies (c).

(c) \implies (d). Let us fix $r > 0$ such that (c) is satisfied and choose a covering of \mathbb{R}^n by the balls $B(x_k, r)$, $k = 1, 2, \dots$, and a partition of unity $\{e_k \mid k = 1, 2, \dots\}$ with $\text{supp } e_k \subset B(x_k, r)$ and with the properties formulated above. Then for any $u \in C_c^\infty(\mathbb{R}^n)$ we obtain from (3.5):

$$h_{a,V}(u, u) = \sum_k h_{a,V}(e_k u, e_k u) - \sum_k |\nabla e_k|^2 |u|^2 \geq (\Lambda u, u),$$

where

$$\Lambda(x) = \sum_k \lambda(B(x_k, r)) e_k^2(x) - \sum_k |\nabla e_k|^2.$$

The first sum tends to $+\infty$ as $x \rightarrow \infty$ due to the condition (c), whereas the second sum is bounded. This implies that $\Lambda(x) \rightarrow +\infty$ as $x \rightarrow \infty$, hence (d) is fulfilled.

The implication (d) \implies (e) is obvious.

(e) \implies (a). Assume that (e) is satisfied with the function $\Lambda(x)$. Let us take $u \in \mathcal{L}$. Then

$$\int_{\mathbb{R}^n} \Lambda(x) |u(x)|^2 dx \leq 1.$$

Therefore,

$$\int_{|x| \geq R} |u(x)|^2 dx \leq \left(\inf_{\{|x| \geq R\}} \{\Lambda(x)\} \right)^{-1} \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

so (a) follows from Lemma 3.3. \square

Now we assume that $V \geq 0$ and proceed to some preparatory material which is needed to prove Theorem 1.2.

Denote temporarily $B_r = B(0, r) \subset \mathbb{R}^n$ and define

$$\|\psi\| = \|\psi\|_{L^2(B_r)} = \left(\int_{B_r} |\psi|^2 dx \right)^{1/2}, \quad \|\psi\|_t = \|\psi\|_{L^2(B_{tr})},$$

where $0 < t \leq 1$. Similarly define

$$\begin{aligned} \|\nabla\psi\| &= \|\nabla\psi\|_{L^2(B_r)} = \left(\int_{B_r} |\nabla\psi|^2 dx \right)^{1/2}, \\ \|\nabla\psi\|_t &= \|\nabla\psi\|_{L^2(B_{tr})}. \end{aligned}$$

Lemma 3.4 *The following estimates hold true for any magnetic potential a :*

$$(3.6) \quad \|\psi\| \leq t^{-n/2} \|\psi\|_t + 2^{n+1} r (1-t) \|\nabla_a \psi\|, \quad t \in [1/2, 1];$$

$$(3.7) \quad \|\psi\|^2 \leq 2t^{-n}\|\psi\|_t^2 + 2^{2n+3}r^2(1-t)^2\|\nabla_a\psi\|^2, \quad t \in [1/2, 1].$$

In these estimates we assume that $\psi \in \text{Lip}(B_r)$.

Proof. With $a = 0$ this Lemma was proved in [31] (for cubes instead of balls) and in [24] (see Lemma 2.8 there). Applying this particular case to $|\psi|$ and using Lemma 3.1, we obtain the desired result. \square

The following Lemma for the case $a = 0$ was proved in [31] (for cubes) and [24] (see Lemma 2.9 there):

Lemma 3.5 *Let us assume that $V \geq 0$. Then*

$$(3.8) \quad \mu(B(x, r)) \leq \lambda(B(x, r)) \leq C_1\mu(B(x, r)) + C_2r^{-2},$$

for any $x \in \mathbb{R}^n$, and any $r > 0$. Here C_1 and C_2 depend only on n ; for example we can take $C_1 = 2^{n+3}(1 + 2^{2n+6})$ and $C_2 = 2^{n+7}$.

Proof. The proof given in [24] works in our case (with arbitrary a) if we replace ∇ by ∇_a , use the Leibniz rule (2.5) and apply Lemma 3.4 above. \square

Proof of Theorem 1.2. Equivalence of (b) (respectively (c)) from Theorem 1.1 to (f) (resp. (g)) from Theorem 1.2 follows from Lemma 3.5 provided we additionally assume that $V \geq 0$. \square

Proof of Theorem 1.3. Without loss of generality we can assume that $V \geq 0$. The necessity part of the theorem follows from Theorem 1.1. (If $\sigma = \sigma_d$ then we can even make $\Lambda(x) \rightarrow +\infty$ as $x \rightarrow \infty$.)

Now let us assume that there exists $\Lambda(x)$ which is semi-bounded below, satisfies (M_c) with some $c > 0$, and $H_{a,V} \geq \Lambda(x)$, i.e.

$$h_{a,V}(u, u) \geq (\Lambda|u|, |u|)$$

for any $u \in \text{Lip}_c(\mathbb{R}^n)$. Using Lemma 3.1 we also obtain

$$h_{a,V}(u, u) \geq h_{a,0}(u, u) \geq h_{0,0}(|u|, |u|).$$

Adding these two inequalities we get

$$2h_{a,V}(u, u) \geq h_{0,\Lambda}(|u|, |u|).$$

Let us introduce the set

$$\tilde{\mathcal{L}} = \{u | u \in \text{Lip}_c(\mathbb{R}^n), h_{0,\Lambda}(u, u) \leq 1\},$$

which has the small tails property (3.4) due to the Molchanov theorem and the Small Tails Lemma 3.3. With \mathcal{L} defined by (3.2) we see that the map $u \mapsto |u|$ maps \mathcal{L} into $\sqrt{2}\tilde{\mathcal{L}}$. Hence \mathcal{L} also has the small tails property and we conclude from Lemma 3.3 that $H_{a,V}$ has a discrete spectrum. \square

Proof of Theorem 1.5. We immediately conclude from the conditions that for any function $u \in \text{Lip}_c(\mathbb{R}^n)$

$$\begin{aligned} h_{a,V}(u, u) &= h_{a,0}(u, u) + (Vu, u) \\ &= (1 - \delta)h_{a,0}(u, u) + \delta h_{a,0}(u, u) + (V|u|, |u|) \\ &\geq (1 - \delta)h_{0,0}(|u|, |u|) + ((\delta\Lambda + V)|u|, |u|) \\ &= (1 - \delta) [h_{0,0}(|u|, |u|) + ((1 - \delta)^{-1}(V + \delta\Lambda)|u|, |u|)]. \end{aligned}$$

Now the same argument based on the Small Tails Lemma 3.3, as in the proof of Theorem 1.3, ends the proof of Theorem 1.5. \square

4 Bounded magnetic field perturbations

The main goal of this section is the proof of the following

Theorem 4.1 *Suppose we are given an electric potential $V \in L_{loc}^\infty(\mathbb{R}^n)$ which is semi-bounded below, and two magnetic potentials $a, \tilde{a} \in \text{Lip}_{loc}(\mathbb{R}^n)$ such that for the magnetic field \tilde{B} , associated with \tilde{a} , we have $\tilde{B} \in \text{Lip}_{loc}(\mathbb{R}^n)$ and*

$$(4.1) \quad |\tilde{B}(x)| \leq C, \quad x \in \mathbb{R}^n.$$

Then $H_{a+\tilde{a},V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum if and only if this is true for $H_{a,V}$.

We will start with the following version of Poincaré Lemma which is similar to the one used by A. Iwatsuka [18], Proposition 3.2.

Lemma 4.2 *Let $B = \sum_{j < k} B_{jk} dx^j \wedge dx^k$ be a closed 2-form in $B(x_0, r) \subset \mathbb{R}^n$ with $B_{jk} \in \text{Lip}(B(x_0, r))$ and $\|B_{jk}\|_\infty \leq C$. Then there exists a 1-form $a = \sum_j a_j dx^j$ with $da = B$, such that $a_j \in \text{Lip}(B(x_0, r))$ and $\|a_j\|_\infty \leq nC$. Here $\|\cdot\|_\infty$ means the L^∞ -norm on $B(x_0, r)$.*

Proof. We can obviously assume that $x_0 = 0$. Then we can produce a_j by the following explicit formulas (see e.g. [44], p. 155–156):

$$a_j(x) = \sum_{k=1}^n x_k \int_0^1 t B_{kj}(tx) dt,$$

and all necessary estimates obviously follow. \square

Proof of Theorem 4.1. Let us assume that $H_{a+\tilde{a},V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum. We will prove that the same holds for $H_{a,V}$. Clearly this is sufficient to prove the theorem.

We can also assume that $V \geq 0$.

Let us choose an arbitrary ball $B(x_0, r)$. We would like to estimate the bottom of the Dirichlet spectrum of $H_{a+\tilde{a},V}$ which we denote, as before, by

$\lambda(B(x_0, r); H_{a+\tilde{a}, V})$. Using the gauge invariance we can then arbitrarily change \tilde{a} in the ball $B(x_0, r)$ as soon as the relation $d\tilde{a} = \tilde{B}$ is preserved. Therefore we can use Lemma 4.2 and assume that $\|\tilde{a}\|_\infty \leq nrC$, where C is the constant in (4.1) and

$$\|\tilde{a}\|_\infty = \max_j \|\tilde{a}_j\|_\infty.$$

Denote

$$P'_j = \frac{1}{i} \frac{\partial}{\partial x^j} + a_j + \tilde{a}_j, \quad P_j = \frac{1}{i} \frac{\partial}{\partial x^j} + a_j.$$

Then for any $u \in \text{Lip}_c(B(x_0, r))$ we have

$$\|P'_j u\|^2 = \|P_j u + \tilde{a}_j u\|^2 = \|P_j u\|^2 + \|\tilde{a}_j u\|^2 + 2\text{Re}(P_j u, \tilde{a}_j u),$$

hence

$$h_{a+\tilde{a}, 0}(u, u) = \sum_{j=1}^n \|P'_j u\|^2 \leq h_{a, 0}(u, u) + n\|\tilde{a}\|_\infty^2 \|u\|^2 + 2 \sum_{j=1}^n \text{Re}(P_j u, \tilde{a}_j u).$$

We have for any $\varepsilon > 0$

$$\begin{aligned} 2\text{Re}(P_j u, \tilde{a}_j u) &\leq 2\|P_j u\| \|\tilde{a}_j u\| \\ &\leq \varepsilon \|P_j u\|^2 + \frac{1}{\varepsilon} \|\tilde{a}_j u\|^2 \leq \varepsilon \|P_j u\|^2 + \frac{1}{\varepsilon} \|\tilde{a}\|_\infty^2 \|u\|^2. \end{aligned}$$

Combining this with the previous estimate we obtain:

$$h_{a+\tilde{a}, 0}(u, u) \leq (1 + \varepsilon)h_{a, 0}(u, u) + n \left(1 + \frac{1}{\varepsilon}\right) \|\tilde{a}\|_\infty^2 (u, u).$$

Now adding (Vu, u) to the left hand side of this inequality and $(1 + \varepsilon)(Vu, u)$ to the right hand side, we obtain the following operator inequality (which holds in the sense of quadratic forms on functions $u \in C_c^\infty(B(x_0, r))$):

$$\begin{aligned} H_{a+\tilde{a}, V} &\leq (1 + \varepsilon)H_{a, V} + n \left(1 + \frac{1}{\varepsilon}\right) \|\tilde{a}\|_\infty^2 \\ &= (1 + \varepsilon) \left(H_{a, V} + \frac{n}{\varepsilon} \|\tilde{a}\|_\infty^2\right) \leq (1 + \varepsilon) \left(H_{a, V} + \frac{n^3 r^2 C^2}{\varepsilon}\right). \end{aligned}$$

It follows that

$$\lambda(B(x_0, r); H_{a+\tilde{a}, V}) \leq (1 + \varepsilon) \left(\lambda(B(x_0, r); H_{a, V}) + \frac{n^3 r^2 C^2}{\varepsilon}\right).$$

Using the IMS localization formula (3.5), we easily conclude that $H_{a, V}$ is semi-bounded below on $C_c^\infty(\mathbb{R}^n)$, hence essentially self-adjoint due to Theorem 2.1.

Now the discreteness of the spectrum for $H_{a, V}$ immediately follows from the localization Theorem 1.1. \square

Corollary 4.3 *Let a be a magnetic potential such that $a \in \text{Lip}_{loc}(\mathbb{R}^n)$ and $B = da$ is also in $\text{Lip}_{loc}(\mathbb{R}^n)$ and bounded. Let also $V \in L_{loc}^\infty(\mathbb{R}^n)$ be a bounded below electric potential. Then $H_{a,V}$ has a discrete spectrum if and only if V satisfies the Molchanov condition (M_c) with some $c > 0$.*

Proof. Theorem 4.1 implies in our case that $H_{a,V}$ has a discrete spectrum if and only if this is true for $H_{0,V}$. By the Molchanov theorem the later is equivalent to the fulfillment of (M_c) (for V) with some $c > 0$. \square

Corollary 4.4 *Let a be a magnetic potential which corresponds to a constant magnetic field. Let also $V \in L_{loc}^\infty(\mathbb{R}^n)$ be a bounded below electric potential. Then $H_{a,V}$ has a discrete spectrum if and only if V satisfies the Molchanov condition (M_c) with some $c > 0$.*

5 Sufficient conditions

5.1 Case $n = 2$

We will start by demonstrating the uncertainty principle argument in the proof of the following well known Lemma (see e.g. [1]):

Lemma 5.1 *Assume that $n = 2$. Then*

$$(5.1) \quad H_{a,0} \geq B(x) \quad \text{and} \quad H_{a,0} \geq -B(x),$$

where B as usual denotes the magnetic field produced by the magnetic potential a . The inequalities (5.1) hold in the sense of operator inequalities (i.e. inequalities of the quadratic forms) on $C_c^\infty(\mathbb{R}^2)$.

Proof. We have $H_{a,0} = P_1^2 + P_2^2$ with $[P_1, P_2] = -iB$. (See notations in Sect. 1.1.) Hence for any $u \in C_c^\infty(\mathbb{R}^2)$

$$\begin{aligned} (Bu, u) \leq |(Bu, u)| &= |((P_1P_2 - P_2P_1)u, u)| = |(P_2u, P_1u) - (P_1u, P_2u)| \\ &= |2\text{Im}(P_1u, P_2u)| \leq 2\|P_1u\|\|P_2u\| \leq \|P_1u\|^2 + \|P_2u\|^2 \\ &= (H_{a,0}u, u). \end{aligned}$$

The second inequality in (5.1) follows from the first one (by changing enumeration of coordinates). \square

Proof of Theorem 1.6. The condition $|B(x)| \rightarrow \infty$ means that either $B \rightarrow +\infty$ or $-B \rightarrow +\infty$. In any of these cases the condition (d) of Theorem 1.1 is satisfied. \square

Corollary 5.2 *For $n = 2$ and any $\delta \in [-1, 1]$ we have*

$$(5.2) \quad H_{a,V} \geq V + \delta B$$

Proof. Using the decomposition $H_{a,V} = H_{a,0} + V$ and Lemma 5.1 we obtain

$$(5.3) \quad H_{a,V} \geq B + V \quad \text{and} \quad H_{a,V} \geq -B + V.$$

Multiplying the first inequality by $\kappa \in [0, 1]$, the second by $1 - \kappa$ and adding we obtain (5.2) after denoting $1 - 2\kappa$ by δ . \square

Corollary 5.3 *Assume that $n = 2$ and there exists $\delta \in [-1, 1]$ such that*

$$(5.4) \quad V(x) + \delta B(x) \rightarrow +\infty \quad \text{as} \quad x \rightarrow \infty.$$

Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Proof. Using Corollary 5.2 we see that the condition (d) of Theorem 1.1 is fulfilled. \square

Corollary 5.4 *Assume that $n = 2$ and $\Omega \subset \mathbb{R}^2$ is an open set. Then for any $\delta \in [-1, 1]$*

$$\lambda(\Omega; H_{a,V}) \geq \inf_{x \in \Omega} \{V(x) + \delta B(x)\}.$$

Proof. We can restrict the estimates given in Lemma 5.1 and Corollary 5.2 to the functions from $C_c^\infty(\Omega)$ for any $\Omega \subset \mathbb{R}^2$. \square

Proof of Theorem 1.9. The result immediately follows from Lemma 5.1 and Theorem 1.5 (with $\Lambda(x) \equiv B(x)$). \square

Remark. V. Ivrii noticed that if we take any $\delta \notin [-1, 1]$ then the result of the Corollary 5.3 does not hold any more. Without loss of generality we can assume that $\delta > 1$. More precisely, we have

Proposition 5.5 (V. Ivrii) *There exists a magnetic Schrödinger operator $H_{a,V}$ with C^∞ potentials a, V in \mathbb{R}^2 such that $H_{a,V} \geq 0$, the spectrum of $H_{a,V}$ is not discrete, but for any $\delta > 1$ we have $V(x) + \delta B(x) \rightarrow +\infty$ as $x \rightarrow \infty$.*

Proof. It is well known from the Landau calculation that the bottom of the spectrum of the operator $H_{a,0}$ with constant magnetic field B is precisely $|B|$ (see e.g. Sect.6.1.1. in [17] for a more general calculation).

Let us take a sequence of constants B_j , $j = 1, 2, \dots$, $B_j > 0$, $B_j \rightarrow +\infty$ as $j \rightarrow \infty$. Then define $V_j = -B_j$ and consider the Schrödinger operators $H_j = H_{a_j, V_j}$, where a_j is a magnetic potential corresponding to the constant magnetic field B_j . Then the bottom of the spectrum of H_j in $L^2(\mathbb{R}^2)$ is 0 for all j .

Note that the bottom of the spectrum of H_j in $L^2(\mathbb{R}^2)$ can be defined as the bottom of the Dirichlet spectrum (see (1.16)) which involves only functions with a compact support. Therefore for any $\varepsilon > 0$ and any $x \in \mathbb{R}^2$ we can find $R_j > 0$ such that $\lambda(B(x, R_j); H_j) < \varepsilon$. In fact this does not depend either on the choice of the potential a_j , or on the center point x due to the gauge invariance.

Let us fix $\varepsilon > 0$ (e.g. take $\varepsilon = 1$).

Let us choose a sequence of points x_j , $j = 1, 2, \dots$, such that the balls $B(x_j, R_j + 1)$ are disjoint. Let us construct a function $B \in C^\infty(\mathbb{R}^2)$ such that $B(x) = B_j$ on $B(x_j, R_j)$ and $B(x) \rightarrow +\infty$ as $x \rightarrow \infty$. By the Poincaré Lemma we can find a magnetic potential $a = a_k dx^k$ in \mathbb{R}^2 with $a_k \in C^\infty(\mathbb{R}^2)$, $k = 1, 2$, such that the corresponding magnetic field is $B(x) dx^1 \wedge dx^2$ (or $B(x)$). Define also $V(x) = -B(x)$ and consider the magnetic Schrödinger operator $H_{a,V}$ with a and V as constructed above.

It follows from Corollary 5.2 (with $\delta = 1$) that $H_{a,V} \geq 0$. On the other hand it is clear from the variational principle for the Dirichlet spectrum that there are infinitely many points of the spectrum of $H_{a,V}$ below 2ε . Therefore the spectrum of $H_{a,V}$ is not discrete. At the same time for any fixed $\delta > 1$ we have $V(x) + \delta B(x) = (\delta - 1)B(x) \rightarrow +\infty$ as $x \rightarrow \infty$. \square

5.2 Case $n \geq 3$

The case $n \geq 3$ is substantially more complicated than the case $n = 2$, in particular because no growth condition on $|B|$ would suffice for a reasonable estimate below for $H_{a,0}$ as was shown by A. Iwatsuka [18]. We will establish however that appropriate regularity conditions on B , such as the ones imposed in [1], [8], [18], can be incorporated in the growth conditions for suitable effective potentials, so that results similar to Theorem 1.9 hold.

5.2.1 The Iwatsuka identity

We will start with the following Lemma ((6.2) on page 370 in [18]):

Lemma 5.6 (IWATSUKA IDENTITY) *Assume that we are given an open set $\Omega \subset \mathbb{R}^n$, a magnetic potential a (with components from $\text{Lip}_{loc}(\Omega)$) and a set of real-valued functions $A_{jk} \in \text{Lip}_{loc}(\Omega)$, $j, k = 1, \dots, n$, $A_{kj} = -A_{jk}$. Then*

$$(5.5) \quad 2 \sum_{k < j} \text{Im}(A_{kj} P_k u, P_j u) = \left(\left[\sum_{k < j} A_{kj} B_{kj} \right] u, u \right) + \sum_{k,j} \left(\frac{\partial A_{kj}}{\partial x^k} P_j u, u \right),$$

for any $u \in W_{comp}^{1,2}(\Omega)$, i.e. $u \in L^2(\Omega)$ such that u has a compact support in Ω and $\nabla u \in (L^2(\Omega))^n$ (in the sense of distributions).

Proof. We reproduce the proof for the sake of completeness and also because we have a different sign convention in the definition of a compared with [18].

An obvious approximation argument shows that it is sufficient to consider $u \in C_c^\infty(\mathbb{R}^n)$. Then using integration by parts and the commutation relations

(1.11) we obtain

$$\begin{aligned}
& 2 \sum_{k < j} \operatorname{Im} (A_{kj} P_k u, P_j u) \\
&= \frac{1}{i} \sum_{k < j} \{ (A_{kj} P_k u, P_j u) - (A_{kj} P_j u, P_k u) \} \\
&= \frac{1}{i} \sum_{k < j} ((P_j A_{kj} P_k - P_k A_{kj} P_j) u, u) \\
&= \frac{1}{i} \sum_{k < j} \left\{ (A_{kj} [P_j, P_k] u, u) + \frac{1}{i} \left(\frac{\partial A_{kj}}{\partial x^j} P_k u, u \right) - \frac{1}{i} \left(\frac{\partial A_{kj}}{\partial x^k} P_j u, u \right) \right\} \\
&= \left(\left[\sum_{k < j} A_{kj} B_{kj} \right] u, u \right) + \sum_{k, j} \left(\frac{\partial A_{kj}}{\partial x^k} P_j u, u \right). \quad \square
\end{aligned}$$

It is natural to consider the functions $A_{jk} = -A_{kj}$ as coefficients of a 2-form which we will call a *dual field* or a *dual form*.

The Iwatsuka identity (5.5) plays a very important role in the arguments below. As A. Iwatsuka noticed in [18], by choosing different dual fields A_{jk} we can obtain different estimates leading to various sufficient conditions for the discreteness of spectrum of the operator $H_{a,0}$. We will develop this idea further by incorporating the electric potential into the picture. In this way different effective potentials V_{eff} emerge such that the Molchanov condition for V_{eff} implies that $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum. We will see that the results of J. Avron, I. Herbst and B. Simon [1], A. Dufresnoy [8] and A. Iwatsuka [18] about the discreteness of spectrum can be extended to the case when both electric and magnetic fields contribute to the localization of the quantum particle.

5.2.2 First choice

The first choice of the dual field which we will discuss, is related to the Dufresnoy sufficiency result [8] and will lead to its generalization which takes into account the electric field. In the situation of [8], considering the operator $H_{a,0}$, we can take

$$(5.6) \quad A_{jk}(x) = \frac{B_{jk}(x)}{|B(x)|},$$

where $|B|$ is defined as in (1.10). (Note, however, that A. Dufresnoy used different arguments in [8].) To be able to use this choice we need to know that $B(x) \neq 0$ for large $|x|$ (i.e. if $|x| \geq R$ where $R > 0$ is sufficiently large. This was irrelevant in the situation when $|B(x)| \rightarrow \infty$ as $x \rightarrow \infty$ and $V = 0$ which was considered in [8]. In a more general situation which will be considered here we

do not want to impose any a priori growth or non-vanishing requirement on B . By this reason we will smooth down the field (5.6) at the places where $|B|$ is small and use the dual field given by (1.24).

We will use the following notation:

$$|\nabla B| = \left(\sum_{j < k} |\nabla B_{jk}|^2 \right)^{1/2}.$$

Similarly we define $|\nabla \beta|$ and $|\nabla A|$ below.

Denote also $\beta_{jk} = B_{jk}/|B|$. Differentiating β_{jk} gives

$$(5.7) \quad \nabla \beta_{jk} = |B|^{-1} \nabla B_{jk} - |B|^{-2} B_{jk} \nabla |B|.$$

Using the inequality $|\nabla |B|| \leq |\nabla B|$, we obtain

$$(5.8) \quad |\nabla \beta_{jk}| \leq |B|^{-1} |\nabla B_{jk}| + |B|^{-2} |B_{jk}| |\nabla B|,$$

hence by the triangle inequality

$$(5.9) \quad |\nabla \beta| \leq 2|B|^{-1} |\nabla B|.$$

Differentiating (1.24) we obtain

$$(5.10) \quad \nabla A_{jk} = \chi(|B|) \nabla \beta_{jk} + \beta_{jk} \chi'(|B|) \nabla |B|,$$

hence

$$|\nabla A_{jk}| \leq \chi(|B|) |\nabla \beta_{jk}| + \chi'(|B|) |\beta_{jk}| |\nabla B|.$$

Therefore for $1/2 \leq |B| \leq 1$ we have

$$(5.11) \quad |\nabla A_{jk}| \leq |\nabla \beta_{jk}| + 2|\beta_{jk}| |\nabla B|$$

and

$$(5.12) \quad |\nabla A| \leq |\nabla \beta| + 2|\nabla B| \leq 2|B|^{-1} |\nabla B| + 2|\nabla B| \leq 6|\nabla B|.$$

Since $A = 0$ for $|B| \leq 1/2$, the estimate (5.12) holds if $|B| \leq 1$.

Now introducing a majorant function

$$(5.13) \quad M_B(x) = \begin{cases} |\nabla \beta_{jk}| & \text{if } |B| \geq 1, \\ 6|\nabla B| & \text{if } |B| < 1, \end{cases}$$

we see that

$$(5.14) \quad |\nabla A(x)| \leq M_B(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Note also that

$$|B|^{-1} \leq 2(1 + |B|)^{-1} \quad \text{if } |B| \geq 1,$$

and

$$|\nabla B| \leq 2(1 + |B|)^{-1} |\nabla B| \quad \text{if } |B| \leq 1.$$

These estimates together imply that

$$M_B(x) \leq 12(1 + |B|)^{-1} |\nabla B|$$

for all x , hence (5.14) gives

$$(5.15) \quad |\nabla A| \leq 12(1 + |B|)^{-1} |\nabla B|$$

for all $x \in \mathbb{R}^n$.

Now we are ready for

Proof of Theorem 1.12. Let us use the Iwatsuka identity (5.5) with A_{kj} given by (1.24). Clearly

$$\sum_{k < j} A_{kj} B_{kj} = \chi(|B|) |B|,$$

and the first term in the right hand side of (5.5) becomes $(\chi(|B|) |B| u, u)$. Since

$$\chi(|B|) |B| \geq |B| - 1, \quad \text{and} \quad |A_{jk}| \leq 1,$$

we obtain, using (1.25), that for any $u \in \text{Lip}_c(\mathbb{R})$:

$$\begin{aligned} (|B|u, u) &\leq 2 \sum_{k < j} (|A_{kj}| |P_k u|, |P_j u|) + \sum_{k, j} \left(\left| \frac{\partial A_{kj}}{\partial x^k} \right| |P_j u|, |u| \right) + (u, u) \\ &\leq (n-1) \sum_{k=1}^n \|P_k u\|^2 + \sum_{j=1}^n (|P_j u|, X|u|) + (u, u) \\ &= (n-1) h_{a,0}(u, u) + \sum_{j=1}^n (|P_j u|, X|u|) + (u, u). \end{aligned}$$

Choosing an arbitrary $\varepsilon > 0$, we see that the middle term in the right hand side is estimated by

$$\varepsilon \sum_{j=1}^n \|P_j\|^2 + \frac{n}{4\varepsilon} (X^2 u, u) = \varepsilon h_{a,0}(u, u) + \frac{n}{4\varepsilon} (X^2 u, u).$$

Therefore we obtain

$$(|B|u, u) \leq (n-1 + \varepsilon) h_{a,0}(u, u) + \frac{n}{4\varepsilon} (X^2 u, u) + (u, u),$$

and

$$h_{a,0}(u, u) \geq \frac{1}{n-1 + \varepsilon} (|B|u, u) - \frac{n}{4\varepsilon(n-1 + \varepsilon)} (X^2 u, u) - \frac{1}{n-1 + \varepsilon} (u, u).$$

Now we can apply Theorem 1.5 with

$$\Lambda(x) = \frac{1}{n-1+\varepsilon}|B| - \frac{n}{4\varepsilon(n-1+\varepsilon)}X^2 - \frac{1}{n-1+\varepsilon},$$

which immediately leads to the desired result. \square

Taking a specific majorant $X(x)$ in Theorem 1.12 we can make the result more specific. In this way we obtain for example the following

Corollary 5.7 *The result of Theorem 1.12 holds if we replace the majorant function X in the definition of the effective potential (1.26) either by $\sqrt{n-1}M_B$ or by $12\sqrt{n-1}(1+|B|^{-1})|\nabla B|$.*

Proof. It is sufficient to notice that

$$\sum_{k=1}^n \left| \frac{\partial A_{kj}}{\partial x^k} \right| \leq \sqrt{n-1}|\nabla A|,$$

due to the Cauchy–Schwarz inequality. \square

We will need a notation for domination between two real-valued functions f, g on $S \subset \mathbb{R}^n$. Namely, we will write

$$(5.16) \quad f \prec g \quad \text{or} \quad g \succ f$$

on S if for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$(5.17) \quad f(x) \leq \varepsilon g(x) + C(\varepsilon) \quad \text{for all } x \in S.$$

It is easy to see that $f \prec g$ and $g \prec h$ imply $f \prec h$. If $f(x) \geq 0$, $g(x) > 0$ for all $x \in S$, f, g are locally bounded and $g(x) \rightarrow +\infty$ as $x \rightarrow \infty$, then the relation (5.16) is equivalent to

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow \infty,$$

i.e. $f(x)/g(x) \rightarrow 0$ as $x \rightarrow \infty$.

Note also that if g is semi-bounded below on S , then

$$(5.18) \quad f \prec g \iff f \prec |g| \iff f \prec 1 + |g|.$$

The proof immediately follows from the implication

$$g \geq -C \implies |g| \leq g + 2C.$$

Now we are ready for the formulation of an important corollary of Theorem 1.12. Note that though this theorem does not explicitly include any regularity requirements on B , they are implicit in the requirements on the effective potential. We will make the result more explicit (though weaker) by invoking some explicit domination requirements on the majorant X .

Corollary 5.8 *Let us assume that there exists $\delta \in [0, 1)$ such that the effective potential*

$$(5.19) \quad V_{\text{eff}}^{(\delta)}(x) = V(x) + \frac{\delta}{n-1}|B(x)|$$

is semi-bounded below and satisfies the Molchanov condition (M_c) with some $c > 0$ (in particular, this holds if $V_{\text{eff}}^{(\delta)}(x) \rightarrow +\infty$ as $x \rightarrow \infty$).

In addition assume that the square of the majorant function $X(x)$ from (1.25) is dominated by $V_{\text{eff}}^{(\delta)}$:

$$(5.20) \quad X^2 \prec V_{\text{eff}}^{(\delta)} \quad \text{on } \mathbb{R}^n.$$

Then the operator $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Remark. Since $V_{\text{eff}}^{(\delta)}$ is bounded below and locally bounded, replacing $V_{\text{eff}}^{(\delta)}$ by $1 + |V_{\text{eff}}^{(\delta)}|$ in (5.20) leads to an equivalent relation.

Proof of Corollary 5.8. Let us choose an arbitrary $\varepsilon > 0$. The condition (5.20) means that for any $\kappa > 0$ there exists $C(\kappa) > 0$ such that

$$(5.21) \quad X^2(x) \leq \kappa V_{\text{eff}}^{(\delta)}(x) + C(\kappa), \quad x \in \mathbb{R}^n.$$

Now for any $\varepsilon > 0$ and $\delta \in [0, 1)$ we can find $\varepsilon > 0$ and $\delta' \in [0, 1)$ such that

$$\frac{\delta'}{n-1+\varepsilon} = \frac{\delta}{n-1},$$

so that

$$V_{\text{eff}}^{(\delta',\varepsilon)}(x) = V_{\text{eff}}^{(\delta)}(x) - \frac{n\delta'}{4\varepsilon(n-1+\varepsilon)}X^2(x)$$

for $V_{\text{eff}}^{(\delta,\varepsilon)}$ as in (1.26). It follows from (5.21) that

$$V_{\text{eff}}^{(\delta',\varepsilon)}(x) \geq (1-\kappa)V_{\text{eff}}^{(\delta)} - C(\delta, \varepsilon, \kappa),$$

hence $V_{\text{eff}}^{(\delta',\varepsilon)}$ is also semi-bounded below and satisfies (M_c) with the same $c > 0$ as for $V_{\text{eff}}^{(\delta)}$. Therefore the desired result follows from Theorem 1.12. \square

Now we will replace the conditions on the majorant X in Corollary 5.8 by an explicit choice of X which is of A. Dufresnoy type [8]. In this way we get a theorem which improves the sufficiency result of [8] adding electric field and capacity to the picture.

Theorem 5.9 *Let us assume that there exists $\delta \in [0, 1)$ such that the effective potential (5.19) is semi-bounded below and satisfies the Molchanov condition (M_c) with some $c > 0$ (in particular, this holds if $V_{\text{eff}}^{(\delta)}(x) \rightarrow +\infty$ as $x \rightarrow \infty$).*

In addition assume that one of the following conditions (a), (b) is satisfied:

$$(a) M_B^2 \prec V_{\text{eff}}^{(\delta)},$$

or, in other words,

$$|\nabla\beta|^2 \prec V_{\text{eff}}^{(\delta)} \quad \text{on} \quad \{x \mid |B(x)| \geq 1\},$$

and

$$|\nabla B|^2 \prec V_{\text{eff}}^{(\delta)}(x) \quad \text{on} \quad \{x \mid |B(x)| < 1\};$$

$$(b) (1 + |B|)^{-2} |\nabla B|^2 \prec V_{\text{eff}}^{(\delta)} \quad \text{on} \quad \mathbb{R}^n.$$

Then the operator $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Proof. According to (5.14) and (5.15) the conditions (a) or (b) imply that we can take the majorant

$$X(x) = M_B(x) \quad \text{or} \quad 12(1 + |B(x)|)^{-1} |\nabla B(x)|$$

respectively, and apply Corollary 5.8. \square

The following even more explicit result also improves the sufficiency result by A. Dufresnoy [8].

Theorem 5.10 *Let us assume that $B_{jk} \in \text{Lip}(\mathbb{R}^n)$ for all j, k , and the following conditions are satisfied:*

$$(5.22) \quad |\nabla B| \leq C \quad \text{if} \quad |B| \leq 1;$$

$$(5.23) \quad |\nabla\beta| = o(|B|^{1/2}) \quad \text{as} \quad |B| \rightarrow \infty.$$

Assume also that there exists $\delta \in [0, 1)$ such that the effective potential (5.19) is semi-bounded below and satisfies the Molchanov condition (M_c) with some $c > 0$. Then the operator $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Proof. The result easily follows from Theorem 5.9 if $V \geq 0$ (or if V is semi-bounded below). In the general case it can be deduced from Theorem 1.12 or Corollary 5.7 by the same argument which was used in the proof of Corollary 5.8, except that in this case instead of (5.21) we should estimate $X^2(x)$ by $\kappa|B(x)| + C(\kappa)$. \square

Proof of Theorem 1.13. To deduce this theorem from Theorem 5.10 it is sufficient to show that the condition (5.23) imposed on β in Theorem 5.10, follows from the condition $(B_{3/2}^0)$ which is imposed on B and means that

$$|\nabla B| = o\left((1 + |B|)^{3/2}\right) \quad \text{as} \quad |B| \rightarrow \infty.$$

This implication immediately follows from the estimate (5.9). \square

Remark 1. It is also possible to prove Theorem 1.13 and all previous theorems of this subsection (except the ones with conditions which explicitly include β_{jk}) by the following choice of the dual field:

$$A_{jk}(x) = \frac{B_{jk}(x)}{\langle B(x) \rangle},$$

where $\langle B \rangle = (1 + |B|^2)^{1/2}$. This choice leads to the arguments and estimates which are similar to the ones used above in the proof of Theorem 1.12.

Remark 2. Sufficient measure conditions similar to Corollaries 1.15 and 1.16 hold for all types of effective potentials discussed above.

5.2.3 Second choice

Now we will discuss the choice of the dual field A_{jk} which is associated with the field suggested by A. Iwatsuka [18]:

$$A_{jk}(x) = \frac{B_{jk}(x)}{|B(x)|^2}.$$

It is proved in [18] that this choice leads to a weakest regularity condition on B which guarantees the discreteness of spectrum for $H_{a,0}$ provided $|B(x)| \rightarrow \infty$ as $x \rightarrow \infty$.

We will improve the result of [18] by adding a term taking into account the electric field. Hence by the same reason as above we will slightly modify this field as follows:

$$(5.24) \quad A_{jk}(x) = \frac{B_{jk}(x)}{\langle B(x) \rangle^2}.$$

We will assume that $B_{jk} \in \text{Lip}_{loc}(\mathbb{R}^n)$ for all j, k . A. Iwatsuka [18] assumes also that the condition (B_2^0) from Sect.1.4 holds. Choosing an arbitrarily small $r > 0$ denote

$$(5.25) \quad \varepsilon_x = \sup_{y \in B(x,r)} \frac{1 + |\nabla B(y)|}{\langle B(y) \rangle^2}.$$

The conditions (1.23) and (B_2^0) together are equivalent to the relation

$$(5.26) \quad \varepsilon_x \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

It is proved in [18] that this implies that $H_{a,0}$ has a discrete spectrum.

We will not a priori require either (1.23) or (B_2^0) to be satisfied, but we will include ε_x above into an effective potential so that possible violation of (5.26) is compensated by the electric field. More precisely, we will prove

Theorem 5.11 *Let us assume that $B_{jk} \in \text{Lip}_{loc}(\mathbb{R}^n)$ for all j, k and define an effective potential*

$$(5.27) \quad V_{\text{eff}}^{(\delta)}(x) = V(x) + \frac{\delta}{n-1} \varepsilon_x^{-1/2} (1 - \varepsilon_x).$$

If there exists $\delta \in [0, 1)$ such that $V_{\text{eff}}^{(\delta)}$ is semi-bounded below and satisfies the Molchanov condition (M_c) with a positive $c > 0$, then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Proof. Note first that due to the triangle inequality

$$|B| \leq \langle B \rangle \leq 1 + |B|,$$

so $\langle B \rangle$ can be replaced by $|B|$ (and vice versa) in the domination relations.

We will again use the Iwatsuka identity (5.5). With the choice (5.24) we have

$$(5.28) \quad \sum_{k < j} A_{kj} B_{kj} = \frac{|B|^2}{\langle B \rangle^2} = \frac{\langle B \rangle^2 - 1}{\langle B \rangle^2} = 1 - \frac{1}{\langle B \rangle^2}.$$

Let us fix $r > 0$ and choose $x \in \mathbb{R}^n$. Then using the inequality

$$|A_{kj}(y)| \leq \langle B(y) \rangle^{-1} \leq \varepsilon_x^{1/2}, \quad y \in B(x, r),$$

we obtain for any $u \in C_c^\infty(B(x, r))$ for the left hand side of (5.5):

$$(5.29) \quad \begin{aligned} 2 \sum_{k < j} \text{Im}(A_{kj} P_k u, P_j u) &\leq 2 \sum_{k < j} (|A_{kj}| |P_k u|, |P_j u|) \\ &\leq 2 \varepsilon_x^{1/2} \sum_{k < j} (|P_k u|, |P_j u|) \leq (n-1) \varepsilon_x^{1/2} h_{a,0}(u, u). \end{aligned}$$

Now let us estimate the last term in (5.5). Using the inequality $|\nabla|B|| \leq |\nabla B|$, we obtain

$$\begin{aligned} |\nabla A_{kj}| &= |\langle B \rangle^{-2} \nabla B_{kj} - 2 B_{kj} \langle B \rangle^{-4} |B| |\nabla|B|| \\ &\leq \langle B \rangle^{-2} |\nabla B_{kj}| + 2 |B_{kj}| \langle B \rangle^{-4} |B| |\nabla B| \\ &\leq \langle B \rangle^{-2} |\nabla B_{kj}| + 2 |B_{kj}| \langle B \rangle^{-3} |\nabla B|. \end{aligned}$$

Therefore by the triangle inequality

$$|\nabla A| \leq 3 \langle B \rangle^{-2} |\nabla B|.$$

Now using the Cauchy-Schwarz inequality we find:

$$\sum_k \left| \frac{\partial A_{kj}}{\partial x^k} \right| \leq \sqrt{n-1} |\nabla A| \leq 3 \sqrt{n-1} \langle B \rangle^{-2} |\nabla B|.$$

Hence we can estimate the last term in (5.5) as follows:

$$\begin{aligned}
& \left| \sum_{k,j} \left(\frac{\partial A_{kj}}{\partial x^k} P_j u, u \right) \right| \leq 3\varepsilon_x \sqrt{n-1} \sum_j (|P_j u|, |u|) \\
& \leq 3\varepsilon_x \sqrt{n-1} \left(\kappa_x \sum_j \|P_j u\|^2 + \frac{n}{4\kappa_x} \|u\|^2 \right) \\
& = 3\varepsilon_x \sqrt{n-1} \left(\kappa_x h_{a,0}(u, u) + \frac{n}{4\kappa_x} \|u\|^2 \right),
\end{aligned}$$

where $u \in C_c^\infty(B(x, r))$ and $\kappa_x > 0$ is arbitrary.

Let us fix $\varepsilon > 0$ and then choose κ_x so that

$$3\varepsilon_x \sqrt{n-1} \kappa_x = \varepsilon \varepsilon_x^{1/2}.$$

This leads to the estimate

$$(5.30) \quad \left| \sum_{k,j} \left(\frac{\partial A_{kj}}{\partial x^k} P_j u, u \right) \right| \leq \varepsilon \varepsilon_x^{1/2} h_{a,0}(u, u) + \frac{9n(n-1)}{4\varepsilon} \varepsilon_x^{3/2} \|u\|^2.$$

Now from the identities (5.5), (5.28) and from the estimates (5.29), (5.30) we obtain for any $u \in C_c^\infty(B(x, r))$

$$(n-1+\varepsilon)h_{a,0}(u, u) \geq \left[(1-\varepsilon_x)\varepsilon_x^{-1/2} - \frac{9n(n-1)\varepsilon_x}{4\varepsilon} \right] (u, u).$$

(Here we used the obvious estimate $\langle B(y) \rangle^{-2} \leq \varepsilon_x$ if $y \in B(x, r)$.) The last estimate gives a lower bound for the bottom of the Dirichlet spectrum of $H_{a,0}$ on any ball $B(x, r')$ with $r' < r$:

$$\lambda(B(x, r'); H_{a,0}) \geq (n-1+\varepsilon)^{-1} \left[\varepsilon_x^{-1/2}(1-\varepsilon_x) - \frac{9n(n-1)\varepsilon_x}{4\varepsilon} \right].$$

If $\varepsilon_x < 1$, we can replace ε_x by 1 in the last term to get

$$(5.31) \quad \lambda(B(x, r'); H_{a,0}) \geq (n-1+\varepsilon)^{-1} \left[\varepsilon_x^{-1/2}(1-\varepsilon_x) - \frac{9n(n-1)}{4\varepsilon} \right].$$

This also obviously holds if $\varepsilon_x \geq 1$.

Now we can apply the IMS localization formula argument (see proof of Theorem 1.1) by considering a finite multiplicity covering of \mathbb{R}^n by balls of radius $r/2$, to conclude that $H_{a,0} \geq \Lambda(x)$ with

$$\Lambda(x) = (n-1+\varepsilon)^{-1} \varepsilon_x^{-1/2} (1-\varepsilon_x) - C,$$

where $C = C(r, n, \varepsilon)$. Then the statement of the Theorem follows from Theorem 1.5. \square

Corollary 5.12 *Let us assume that B satisfies the Iwatsuka conditions (1.23) and (B_2^0) (or, equivalently, $\varepsilon_x \rightarrow 0$ as $x \rightarrow \infty$). Then the statement of the Theorem 5.11 holds with the effective potential*

$$V_{\text{eff}}^{(\delta)}(x) = V(x) + \frac{\delta}{n-1} \varepsilon_x^{-1/2}.$$

Remark 1. If $V = 0$ then this Corollary immediately implies the Iwatsuka result [18] about the sufficient condition of the discreteness of the spectrum because then the Molchanov condition (M_c) for the function $x \mapsto \varepsilon_x^{-1/2}$ is fulfilled automatically.

Remark 2. If $|B(x)| \rightarrow \infty$ as $x \rightarrow \infty$ and $|B(x)|$ varies sufficiently slowly (e.g. if $|\nabla|B|| \prec |B|$, which in turn holds e.g. if B satisfies the condition (B_1^0)), then $\varepsilon_x^{-1/2}$ becomes equivalent to $|B(x)|$, so the results of Sect.5.2.3 agree with the results of Sect.5.2.2.

5.2.4 Third choice

Here we will discuss the choice of the dual field A_{jk} which leads to a generalization of the result by J. Avron, I. Herbst and B. Simon [1]. In this choice A_{jk} are constants, though we argue on balls of a fixed radius $r > 0$ and these constants may also depend on the choice of the ball.

The advantage of the result obtained in this way is that no local smoothness of B is required. Even the continuity of B is not needed, though we will still maintain the requirement $a_j \in \text{Lip}_{loc}(\mathbb{R}^n)$, hence $B_{jk} \in L_{loc}^\infty(\mathbb{R}^n)$.

Let us choose a finite multiplicity covering of \mathbb{R}^n by balls $B(\gamma, r)$, $\gamma \in \Gamma$. (For example Γ may be an appropriate lattice in \mathbb{R}^n .) Then for any $\gamma \in \Gamma$ chose $A_{kj}(\gamma) = -A_{jk}(\gamma)$ such that $|A(\gamma)| = 1$ and denote

$$(5.32) \quad r_\gamma = \inf_{y \in B(\gamma, r)} \sum_{k < j} A_{kj}(\gamma) B_{kj}(y).$$

We are interested to make the numbers r_γ as big as possible. Clearly

$$r_\gamma \leq \sup_{y \in B(\gamma, r)} |B(y)|,$$

and this inequality is close to equality if B is almost constant in $B(\gamma, r)$ and we have chosen $A_{kj}(\gamma) = B_{kj}(\gamma)/|B_{kj}(\gamma)|$.

It is proved in [1] that if we can choose $A_{kj}(\gamma)$ so that

$$(5.33) \quad r_\gamma \rightarrow \infty \quad \text{as} \quad \gamma \rightarrow \infty,$$

then the operator $H_{a,0}$ has a discrete spectrum. We will improve this result by adding an electric field into the picture. Denote

$$(5.34) \quad r(x) = \min\{r_\gamma \mid |x - \gamma| \leq r\}.$$

Theorem 5.13 *Denote*

$$(5.35) \quad V_{\text{eff}}^{(\delta)}(x) = V(x) + \frac{\delta}{n-1} r(x),$$

and assume that there exists $\delta \in [0, 1)$ such that the effective potential (5.35) is semi-bounded below and satisfies the Molchanov condition (M_c) with some $c > 0$. Then the operator $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Proof. Let us use the Iwatsuka identity (5.5) for $u \in C_c^\infty(B(\gamma, r))$ and with $A_{kj}(x) = A_{kj}(\gamma)$. The last term in (5.5) vanishes. The first term in the right hand side of (5.5) is estimated from below by $r_\gamma(u, u)$ and the left hand side is estimated from above by $(n-1)h_{a,0}(u, u)$ (see arguments given in the proof of Theorem 1.12). Therefore we obtain:

$$(5.36) \quad h_{a,0}(u, u) \geq \frac{r_\gamma}{n-1}(u, u), \quad u \in C_c^\infty(B(\gamma, r)).$$

Now using the IMS localization formula as in the proof of Theorem 5.11 we conclude that

$$H_{a,0} \geq \frac{r(x)}{n-1} - C,$$

and it remains to apply Theorem 1.5. \square

6 Operators on manifolds

Let (M, g) be a Riemannian manifold (i.e. M is a C^∞ -manifold, (g_{jk}) is a Riemannian metric on M), $\dim M = n$. We will always assume that M is connected. We will also assume that we are given a *positive smooth measure* $d\mu$, i.e. a measure which has a C^∞ positive density $\rho(x)$ with respect to the Lebesgue measure $dx = dx^1 \dots dx^n$ in any local coordinates x^1, \dots, x^n , so we will write $d\mu = \rho(x)dx$. This measure may be completely independent of the Riemannian metric, but may of course coincide with the canonical measure $d\mu_g$ induced by the metric (in this case $\rho = \sqrt{g}$ where $g = \det(g_{jk})$, so locally $d\mu_g = \sqrt{g}dx$).

Denote $\Lambda_{(k)}^p(M)$ the set of all k -smooth (i.e. of the class C^k) complex-valued p -forms on M . We will write $\Lambda^p(M)$ instead of $\Lambda_{(\infty)}^p(M)$. A *magnetic potential* is a real-valued 1-form $a \in \Lambda_{(1)}^1(M)$. So in local coordinates x^1, \dots, x^n it can be written as in (1.6) where $a_j = a_j(x)$ are real-valued C^1 -functions of the local coordinates.

The usual differential can be considered as a first order differential operator

$$d : C^\infty(M) \longrightarrow \Lambda^1(M).$$

The deformed differential (compare (1.12))

$$d_a : C^\infty(M) \longrightarrow \Lambda_{(1)}^1(M), \quad u \mapsto du + iua,$$

is also well defined.

The Riemannian metric (g_{jk}) and the measure $d\mu$ induce an inner product in the spaces of smooth forms with compact support in a standard way. The corresponding completions are Hilbert spaces which we will denote $L^2(M)$ for functions and $L^2\Lambda^1(M)$ for 1-forms. These spaces depend on the choice of the metric (g_{jk}) and the measure $d\mu$. However we will skip this dependence in the notations of the spaces for simplicity of notations.

The corresponding local spaces will be denoted $L^2_{loc}(M)$ and $L^2_{loc}\Lambda^1(M)$ respectively. These spaces do not depend on the metric or measure.

Formally adjoint operators to the differential operators with sufficiently smooth coefficients are well defined through the inner products above. In particular, we have an operator

$$d_a^* : \Lambda^1_{(1)}(M) \longrightarrow C(M),$$

defined by the identity

$$(d_a u, \omega) = (u, d_a^* \omega), \quad u \in C_c^\infty(M), \quad \omega \in \Lambda^1_{(1)}(M).$$

(Here $C_c^\infty(M)$ is the set of all C^∞ functions with compact support on M .)

Therefore we can define the magnetic Laplacian Δ_a (with the potential a) by the formula

$$-\Delta_a = d_a^* d_a : C^\infty(M) \longrightarrow C(M).$$

Now the *magnetic Schrödinger operator* on M is defined as

$$(6.1) \quad H = H_{a,V} = -\Delta_a + V,$$

where $V \in L^\infty_{loc}(M)$, i.e. V is a locally bounded measurable function which is called *the electric potential*. We will always assume V to be real-valued. Then H becomes a symmetric operator in $L^2(M)$ if we consider it on the domain $C_c^\infty(M)$.

Note that for $a = 0$ the operator Δ_a becomes a generalized Laplace-Beltrami operator Δ on scalar functions on M and it can be locally written in the form

$$(6.2) \quad \Delta u = \frac{1}{\rho} \frac{\partial}{\partial x^j} (\rho g^{jk} \frac{\partial u}{\partial x^k}).$$

The following expressions for $H_{a,V}$ are also useful ([36]):

$$(6.3) \quad H_{a,V} u = -\Delta u - 2i\langle a, du \rangle + (id^* a + |a|^2)u + Vu,$$

and in local coordinates

$$(6.4) \quad H_{a,V} u = -\frac{1}{\rho} \left(\frac{\partial}{\partial x^j} + ia_j \right) \left[\rho g^{jk} \left(\frac{\partial}{\partial x^k} + ia_k \right) u \right] + Vu,$$

or in slightly different notations

$$(6.5) \quad H_{a,V}u = \frac{1}{\rho}P_j[\rho g^{jk}P_k u] + Vu = \frac{1}{\rho}(D_j + a_j)[\rho g^{jk}(D_k + a_k)u] + Vu,$$

where $D_j = -i\partial/\partial x_j$.

Now we need a condition on the Riemannian manifold (M, g) which would allow extending the results above to a more general context. This is the condition of *bounded geometry* which means that the injectivity radius r_{inj} is positive and all covariant derivatives of the curvature tensor R are bounded:

- (a) $r_{inj} > 0$;
- (b) $|\nabla^m R| \leq C_m; m = 0, 1, \dots$

Here the norm $|\cdot|$ of tensors $\nabla^m R$ is measured with respect to the given Riemannian metric g . (See more details about these conditions and their use in [24].)

We will also impose a bounded geometry condition on the measure $d\mu$. We will say that *the triple* $(M, g, d\mu)$ *has bounded geometry* if (M, g) is a manifold of bounded geometry, and for a small $r > 0$ in local geodesic coordinates in any ball $B(x, r)$ we have $d\mu = \rho(x)dx$ where $\rho \geq \varepsilon > 0$ and for any multiindex α we have $|\partial^\alpha \rho| \leq C_\alpha$ with the constants ε, C_α independent of x . In particular this automatically holds for the Riemannian measure $d\mu = \sqrt{g}dx$ if (M, g) has bounded geometry.

Some conditions at infinity are needed to guarantee that the operator $H_{a,V}$ is essentially self-adjoint in $L^2(M)$ - see e.g. M. Shubin [36, 37] and references there for such conditions. The essential self-adjointness result by A. Iwatsuka [19] can also be extended to manifolds of bounded geometry. The result which is of particular importance for us is essential self-adjointness of any semi-bounded below magnetic Schrödinger operator on any complete Riemannian manifold (see [37]), in particular on any manifold of bounded geometry.

For a triple $(M, g, d\mu)$ of bounded geometry we can define a capacity of a compact set $F \subset B(x, r)$ for a small $r > 0$ by use of geodesic coordinates or by use of norms induced by the metric g and the measure $d\mu$. Using geodesic coordinates for different balls $B(x, r)$ and $B(x', r)$ to measure $\text{cap}(F)$ for $F \subset B(x, r) \cap B(x', r)$ leads to equivalent results, so it does not affect our formulations. Using Riemannian norms of tensors in the definition of capacity also leads to an equivalent result.

After these explanations all results formulated above make sense and can be extended to the triples of bounded geometry with minor changes (some constants depending on geometry appear in the formulations). In case when $a = 0$ and with the Riemannian measure $d\mu$ this was done by the authors in [24]. We will give examples of such extensions in Sect.6.

Let $(M, g, d\mu)$ be a triple with bounded geometry. Let us chose a ball $B(x_0, r)$ of a fixed sufficiently small radius $r > 0$, and let x^1, \dots, x^n be local geodesic coordinates in this ball.

Lemma 6.1 *Under the conditions above there exists $C = C(M, g, d\mu)$ such that*

$$(6.6) \quad C^{-1} \sum_{j=0}^n \|P_j u\|_0^2 \leq h_{a,0}(u, u) \leq C \sum_{j=0}^n \|P_j u\|_0^2, \quad u \in C_c^\infty(B(x_0, r)),$$

where P_j are defined by (1.2) in the geodesic coordinates, and $\|\cdot\|_0$ means the norm in $L^2(B(x_0, r); dx^1 \dots dx^n)$.

Proof. The result immediately follows from the local presentation (6.5) for $H_{a,V}$ and from the bounded geometry requirements. \square

This Lemma allows an easy extension of all local estimates in the balls of a fixed small radius $r > 0$ to the case of the triples $(M, g, d\mu)$ of bounded geometry. Then we need to use extensions of the localization results from Sect.3. They can indeed be extended due to Gromov's observation on the existence of good coverings [13] and the manifolds extension of the IMS localization formula (3.5) – see the details in [34, 35, 24].

Now we can formulate the simplest result for 2-dimensional manifolds.

Theorem 6.2 *Assume that we have a bounded geometry triple $(M, g, d\mu)$ with $n = \dim M = 2$. Then there exists $\delta_0 > 0$ depending on the triple $(M, g, d\mu)$, such that the following holds.*

Let $H_{a,V}$ be a magnetic Schrödinger operator on M and there exists $\delta \in (-\delta_0, \delta_0)$ such that the effective potential given by (1.21) with B identified with the ratio $B/d\mu$, is semi-bounded below and satisfies the Molchanov condition (M_c) with some $c > 0$. Then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

Proof. The estimate $H_{a,0} \geq \delta B$ follows from Lemmas 6.1 and 5.1 on balls $B(x, r)$. Then the IMS-localization formula (3.5) used for a finite multiplicity covering of M by such balls (of the same radius) leads to a global estimate of the form

$$H_{a,0} \geq \delta B - C,$$

where $C = C(M, g, d\mu)$. Now the desired result immediately follows from the manifold version of Theorem 1.5. \square

Let us formulate an extension of Theorem 1.12 to manifolds.

Let us define a smoothed direction of the magnetic field as a 2-form (or a skew-symmetric (0,2)-tensor):

$$(6.7) \quad A(x) = \chi(|B(x)|) \frac{B(x)}{|B(x)|},$$

where $\chi \in \text{Lip}([0, \infty))$, $\chi(r) = 0$ if $r \leq 1/2$, $\chi(r) = 1$ if $r \geq 1$, $\chi(r) = 2r - 1$ if $1/2 \leq r \leq 1$, so $0 \leq \chi(r) \leq 1$ and $|\chi'(r)| \leq 2$ for all r . The norm $|B|$ is measured by the use of the given Riemannian metric g .

Theorem 6.3 *Let us assume that $B \in \text{Lip}_{loc}(\mathbb{R}^n)$, A is defined by (6.7), and a positive measurable function $X(x)$ in \mathbb{R}^n satisfies*

$$(6.8) \quad |\nabla A(x)| \leq X(x), \quad x \in M,$$

where ∇ means the covariant derivative. Then there exist constants $\delta_0 > 0$ and $C_0 > 0$ depending only on the triple $(M, g, d\mu)$, such that the following is true. If there exists $\delta \in [0, \delta_0)$ such that the effective potential

$$(6.9) \quad V_{\text{eff}}(x) = V(x) + \delta|B(x)| - C_0\delta X^2(x)$$

is bounded below and satisfies the Molchanov condition (M_c) with some $c > 0$ (in particular, this holds if $V_{\text{eff}}(x) \rightarrow +\infty$ as $x \rightarrow \infty$), then $H_{a,V}$ is essentially self-adjoint, semi-bounded below and has a discrete spectrum.

In case $B = 0$ and V semi-bounded below the condition on V in this theorem becomes necessary and sufficient due to the extension of Molchanov theorem given in [24].

Proof of Theorem 6.3 is similar to the proof of Theorem 6.2. Other results from Sect.5 have similar extensions as well.

7 Other results

A review of other results on the discreteness of spectrum of the Schrödinger operators and related topics can be found in [24]. Here we will restrict ourselves to a few specific remarks concerning magnetic Schrödinger operators.

Cwickel-Lieb-Rozenblum (CLR) type estimates for the number of negative eigenvalues for magnetic Schrödinger operators have been proved by J. Avron, I. Herbst and B. Simon [1] (see also [40]), though with the right hand side independent of the magnetic field. The proof used heat kernel estimates based on the Feynman–Kac formula as in the paper by E. Lieb [27], and also the diamagnetic inequality (see [41], [14] or [7], Sect.1.3). An analytic proof was provided by M. Melgaard and G. Rozenblum [30].

The CLR estimates used for $H_{a,V} - M$ for arbitrary $M > 0$ obviously imply sufficient conditions for the discreteness of spectrum (namely, the finiteness of the right hand sides of these estimates for all M). However, the above mentioned results still do not allow to take into account any interaction between the electric and magnetic fields.

Under some stronger conditions on the fields it is possible to obtain even asymptotics for the counting function $N(\lambda; H_{a,V})$ for the eigenvalues of $H_{a,V}$. One of the first results of this kind is due to Y. Colin de Verdière [4]. Numerous further results on such asymptotics can be found in (or extracted from) the book of V. Ivrii [17] (see also references there).

The Lieb–Thirring inequalities give explicit estimates for sums of powers of the negative eigenvalues, or, in other words, for the l^p norms of the sequence of these eigenvalues. If $p = \infty$ this means an estimate for the number of negative

eigenvalues as in the case of the CRL estimate. Under some conditions Lieb–Thirring type inequalities were obtained for $H_{a,V}$ by L. Erdős [10] and for a similar Pauli operator by A.V. Sobolev [43].

A Feynman type estimate for $\text{Tr}(\exp(-tH_{a,V}))$ in explicit purely classical terms was obtained by J.M. Combes, R. Schrader and R. Seiler [5].

There exists a useful interaction between capacities and the Feynman–Kac formula. It was discussed e.g. in the books by I. Chavel [3], K. Ito and H. McKean [16], M. Kac [20] and B. Simon [40]. M. Kac and J.-M. Luttinger [21] noticed an interesting relation between the scattering length and capacity.

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