

FEDOSOV MANIFOLDS

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1. Introduction.

A. Connection (or parallel transport) is the main object in the geometry of manifolds. Indeed, connections allow comparison between geometric quantities associated with different (distant) points of the manifold. In 1917 T. Levi-Civita [Lv] introduced a notion of connection for a manifold embedded into \mathbb{R}^n . In 1918 H. Weyl [We] introduced general symmetric linear connections in the tangent bundle. In 1922 E. Cartan [C1], [C2], [C3] studied non-symmetric linear connections (or affine connections), where the torsion is interpreted in general relativity as the density of intrinsic angular momentum (spin) of matter. Actually Cartan explains that affine connection and linear connection with torsion is the same thing. According to Cartan, connection is a mathematical alias for an observer traveling in space-time and carrying measuring instruments. Global comparison devices (like a transformation group in the Erlangen program by F. Klein) are forbidden by the finite propagation speed restriction (see, for example, [S]).

It might seem that the Riemannian geometry only requires a Riemannian metric, but this is due to the fact that there is a canonical Levi-Civita connection associated with this metric.

In this paper we study geometry of *symplectic connections*, i.e. symmetric (torsion-free) connections which preserve a given symplectic form.

We start (in Sect.1) with a more general context. Let us consider an *almost symplectic manifold*, i.e. manifold M with a non-degenerate exterior 2-form ω (not necessarily closed) and try to describe all linear connections (i.e. connections in the tangent bundle) which preserve this form. In the classical case of non-degenerate symmetric form $g = (g_{ij})$ (Riemannian or pseudo-Riemannian) the answer is well known due to T. Levi-Civita, H. Weyl and E. Cartan: for any tensor $T = (T_{ij}^k)$, which is antisymmetric with respect to the subscripts i, j , there exists a unique connection $\Gamma = (\Gamma_{ij}^k)$ with the torsion tensor

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T (i.e. $\Gamma_{ij}^k - \Gamma_{ji}^k = T_{ij}^k$) such that Γ preserves g . If $T = 0$, this connection is called the *Levi-Civita connection*. It is the *canonical* connection associated with g .

We first prove a skew-symmetric analog of this Levi-Civita theorem. Namely, there is a canonical one-one correspondence between the following two sets of linear connections on an almost symplectic manifold (M, ω) :

- 1) all symmetric connections on M ;
- 2) all connections on M which preserve ω .

This correspondence is obtained as follows: for a connection Γ which preserves ω , we just take its symmetric part $\Pi = (\Pi_{jk}^i)$, $\Pi_{jk}^i = \frac{1}{2}(\Gamma_{jk}^i + \Gamma_{kj}^i)$ (which is again a connection). This gives a bijection between all connections preserving ω and all symmetric connections. The inverse operator is also given by explicit formulas (see (1.7) below).

Note the following analogy with the Levi-Civita construction. There we can prescribe the antisymmetric part of the connection, which is a tensor (the torsion tensor). In particular we can take this tensor to be zero (then we arrive to the Levi-Civita connection). In symplectic case (or rather in case of any non-degenerate antisymmetric form) we can prescribe the symmetric part of the connection which is not a tensor but a symmetric connection.

Geometry of almost symplectic manifolds was studied by H.-C. Lee [Le1], [Le2], P. Libermann [Li1-Li3], V.G. Lemlein [Lm1-Lm4], Yu.I. Levin [Lvn], Ph. Tondeur [To] and I. Vaisman [Va1] (see also references in [Li2, Li3], [Lm4], [Va1]). In particular, V.G. Lemlein and Ph. Tondeur independently gave two different (but less geometric) descriptions of all connections preserving ω . (See also [Va1] for a proof and a refinement of the Ph. Tondeur result.)

V.G. Lemlein [Lm1] also introduced almost symplectic manifolds with a symmetric connection such that $\nabla\omega = \mu(x)d\omega$ (as tensors). It follows that $\mu(x) = 1/3$. He described the connections ∇ with this property in terms of an object γ_{ijk} which is completely symmetric in i, j, k (though it is neither a tensor nor a connection). However the equality $\gamma_{ijk} = 0$ has an invariant meaning for locally projective manifolds. Therefore a locally projective almost symplectic manifold has a canonical symmetric connection (see [Lm3] and also [Lm4], Theorem 45).

B. In Sect.2 we introduce our main object: *Fedosov manifolds*. By definition a Fedosov manifold is a triple (M, ω, Γ) where (M, ω) is a symplectic manifold (i.e. ω is a symplectic form, a non-degenerate closed exterior 2-form, on a C^∞ -manifold M) and Γ is a symplectic connection on M .

The famous result of Fedosov gives a canonical deformation quantization on such a manifold. This quantization is canonically defined by the data (M, ω, Γ) (see e.g. [F1], [F2], [DW-L], [W], [D]).

For the role of symplectic connections in geometric quantization see e.g. [Lch], [Hs]. A symplectic connection can be uniquely associated to a Lagrangian polarization – see [Hs], [M], [Va1]. B. Kostant [K] shows that the cotangent bundle to a Lie group has a canonical

symplectic connection i.e. a canonical structure of a Fedosov manifold. K. Habermann [Ha] defined and studied symplectic Dirac operators on Fedosov manifolds with an additional metaplectic structure.

Note that symplectic connections exist on any symplectic manifold. Indeed, locally such a connection can be constructed in Darboux coordinates by assigning all Christoffel symbols to be identically 0. Then we can glue a global symplectic connection by a partition of unity. Therefore a structure of a symplectic manifold can be always extended to a structure of a Fedosov manifold.

In fact the symplectic connections on a symplectic manifold form an infinite-dimensional affine space with difference vector-space isomorphic to the space of all completely symmetric 3-covariant tensors - see e.g. [BFFLS], [Ve] and Section 1 (the paragraph after (1.5)) below.

Fedosov manifolds constitute a natural generalization of Kähler manifolds. A Kähler manifold can be defined as a Fedosov manifold (M, ω, Γ) with an additional almost complex structure J (in the tangent bundle TM), such that the formula

$$(0.1) \quad g(X, Y) := \omega(JX, Y), \quad X, Y \in T_x M, \quad x \in M,$$

defines a Riemannian metric on M , and Γ is the Levi-Civita connection of this metric. (See e.g. [K-N], vol.2, Ch.IX, Sect.4, or [Br], p.148.)

On the other hand there are symplectic manifolds such that they do not admit any Kähler structure, even with a different ω (see e.g. [Br], p.147, for a Thurston example).

We study the curvature tensor of a symplectic connection (on a Fedosov manifold). The usual curvature tensor R^i_{jkl} is well defined by the connection alone. But the tensor $R_{ijkl} = \omega_{ip} R^p_{jkl}$ is more interesting. (We use the standard Einstein summation convention, so here the summation over p is understood.) This tensor is well known. For example, V.G. Lemlein ([Lm1], [Lm2]) introduced and studied it in a more general context of almost symplectic manifolds with a connection. In particular, some important symmetry properties of this tensor were obtained in [Lm2] and later rediscovered by I. Vaisman [Va1]. Since we tried to make this paper self-contained, we give (usually new) proofs of some properties of this tensor which were first obtained in [Lm2] and [Va1]. Following I.Vaisman [Va1] we will call R_{ijkl} *the symplectic curvature tensor*.

The symplectic curvature tensor has symmetries which are different from the symmetries of the curvature tensor on a Riemannian manifold. Namely, aside of standard symmetries $R_{ijkl} = -R_{ijlk}$, $R_{i(jjk)} = 0$ (here the parentheses mean that we should take sum over all cyclic permutations of the subscripts inside the parentheses), it is symmetric with respect to the first two subscripts: $R_{ijkl} = R_{jikl}$. We deduce from these identities that $R_{(ijkl)} = 0$.

C. In Sect.3 we consider the Ricci tensor $K_{ij} = R^k_{ikj} = \omega^{kp} R_{pikj}$ which is again defined by the connection alone. I. Vaisman [Va1] proved that it is in fact symmetric (as in the case of Riemannian manifolds) and the structure of the Fedosov manifold plays

important role in the proof. We give two proofs of the symmetry of K_{ij} (in particular we deduce it from the identity $R_{(ijkl)} = 0$).

There are two natural non-trivial contractions of the curvature tensor on a Fedosov manifold seemingly leading to two different Ricci type tensors (tensors of type (2,0) or bilinear forms on each tangent space). However we prove that they only differ by a constant factor 2: $\omega^{kl}R_{ijkl} = 2K_{ij}$. We also prove that the operator

$$L = \omega^{ij}\nabla_i\nabla_j ,$$

on vector fields has order zero and corresponds to the Ricci tensor (with one subscript lifted by ω).

In the Riemannian case contracting the Ricci tensor we obtain the scalar curvature. On a Fedosov manifold we can only use the symplectic form for the contraction, and this obviously leads to a trivial result.

Note also that I. Vaisman [Va1] provided a complete group-theoretic analysis of the space of all curvature-type tensors (i.e. tensors with the symmetries described above) as a representation space of the group $\mathrm{Sp}(2n)$, where $2n = \dim_{\mathbb{R}} M$. He proved that the corresponding representation space splits into a direct sum of two subspaces, one of them is the space of all tensors with vanishing Ricci curvature part, and another is also explicitly described. Each of these two subspaces is irreducible.

Vanishing of the Ricci tensor provides the vacuum Einstein equations in the Riemannian or pseudo-Riemannian case. This is a determined system for the components of the metric. We can also write the vacuum Einstein equations $K_{ij} = 0$ on a Fedosov manifold. We call the manifolds with this property *symplectic Einstein manifolds*. (They are called Ricci flat in [Va1].) However $K_{ij} = 0$ is an undetermined system (for the symplectic form together with the symplectic connection). The obvious local gauge transformations (symplectic transformations of coordinates) are not sufficient to remove the underdeterminacy even in dimension 2.

We also give a definition of sectional curvature on a Fedosov manifold. It is a function on non-isotropic 2-planes Π in the tangent bundle, and it is defined as the canonical form of the quadratic form $Z \mapsto R(Z, Z, Y, X)$, $Z \in \Pi$, where $\{X, Y\}$ is a basis of Π such that $\omega(X, Y) = 1$. In other words the sectional curvature has values in the orbit space of the action of $SL_2(\mathbb{R})$ on the set of all quadratic forms on \mathbb{R}^2 . If we denote z_1, z_2 the coordinates of Z in the basis X, Y , then the list of all possible canonical forms looks as follows:

- (a) $r(z_1^2 + z_2^2)$ where $r \in \mathbb{R}$;
- (b) $r(z_1^2 - z_2^2)$ where $r > 0$;
- (c) $\pm z_1^2$.

In particular the standard 2-sphere considered as a Kähler (hence Fedosov) manifold presents the case (a) with $r = 1$, the standard hyperbolic (Lobachevsky) plane gives also the case (a) with $r = -1$. We also give examples of translation invariant Fedosov structures on 2-torus where the sectional curvature is (b) (with a constant $r > 0$) or (c).

D. In Sect.4 we introduce *normal coordinates* on a Fedosov manifold. They are defined as affine coordinates in the tangent space to $T_p M$ which are transferred to M by the exponential map of the given connection Γ at p . Since they are defined by the connection only, the construction does not differ from the Riemannian case. We can also define affine *extensions* of an arbitrary tensor $T = (T_{j_1 \dots j_l}^{i_1 \dots i_k})$ which are tensors $T_{j_1 \dots j_l \alpha_1 \dots \alpha_r}^{i_1 \dots i_k}$ obtained at $p \in M$ by taking the corresponding derivatives of the components of T in normal coordinates centered at p and then evaluating the result at p . Applying this procedure to the Christoffel symbols Γ_{ijk} , we obtain so called *normal tensors*

$$A_{ijk} = 0, \quad A_{ijk\alpha_1}, \quad A_{ijk\alpha_1\alpha_2}, \dots$$

(They are tensors in spite of the fact that Γ_{ijk} do not form a tensor, and due to the fact that Γ_{ijk} undergo the tensor transformation law if the transformation of the coordinates is linear.)

Extending results of T.Y. Thomas and O. Veblen (see e.g. [E, Th, V]) we describe all algebraic relations for the normal tensors and the extensions of the symplectic form at a point of a Fedosov manifold. Compared with the classical situation (arbitrary symmetric connection) a new relation is added to provide the compatibility of the normal tensors with the existence of the preserved symplectic form (see (4.18)). We also prove that all local invariants of a Fedosov manifold are appropriate functions of the components of the normal tensors and of the components of the symplectic form.

E. In Section 5 we study relations between the normal tensors and the curvature tensor. Here we again use an appropriately modified technique by T.Y. Thomas and O. Veblen. We provide formulas which express normal tensors through the extensions of the curvature tensor (or its covariant derivatives) and vice versa. A simplest corollary of these formulas is

$$\omega_{ij,kl} = \frac{1}{3} R_{kl ij} ,$$

where $\omega_{ij,kl}$ is the 2nd extension of ω and R is the curvature tensor.

We also prove that the above mentioned symmetries of the curvature tensor together with the Bianchi identity (see (5.25)) and an integrability identity (see (5.26)) provide a complete system of identities for the curvature tensor and its first covariant derivatives at a point of a Fedosov manifold.

It follows from the results of Sect.5 that any local invariant of a Fedosov manifold is an appropriate function of the components of ω and of the covariant derivatives of the curvature tensor (Theorem 5.11). A complete description of such functions in Darboux coordinates was done by D.E. Tamarkin [Ta]. We plan to discuss local invariants in more detail in a future paper.

F. The Poisson manifolds are a natural generalization of symplectic manifolds (see e.g. [Va2]). They only have ω^{ij} which can be degenerate, but satisfy a natural identity which is equivalent to the Jacobi identity for the corresponding Poisson bracket. Recently M. Kontsevich [Ko] constructed a deformation quantization on any such manifold. Note however that a connection preserving the Poisson structure does not always exist. Indeed,

if it exists then the matrices ω^{ij} should have a locally constant rank, because the parallel transport preserves the rank. This is not the case e.g. if all the components $\omega^{ij}(x)$ are linear in x and not identically 0, as is the case for the linear Poisson brackets defined by non-abelian Lie algebras.

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1. Connections on almost symplectic manifolds.

Suppose we are given a manifold M , $\dim M = n$, and a non-degenerate 2-form ω which we shall fix. Such a pair (M, ω) is called *almost symplectic manifold* (see e.g. [Li] and references there).

Let ∇ be a connection (covariant derivative) on M . We will say that ∇ *preserves* ω if $\nabla\omega = 0$ or

$$(1.1) \quad Z(\omega(X, Y)) = \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y)$$

for any vector fields X, Y, Z .

Let us write this in coordinates. If x^1, \dots, x^n are local coordinates, denote $\partial_i = \partial/\partial x^i$. The components of ω in these coordinates are $\omega_{ij} = \omega(\partial_i, \partial_j)$. Let us denote $\nabla_i = \nabla_{\partial_i}$ and introduce the *Christoffel symbols* Γ_{ij}^k by $\nabla_i \partial_j = \Gamma_{ij}^k \partial_k$. (Here and later on we omit the sign of summation using the standard Einstein convention.) We will identify the connection ∇ with the set $\Gamma = (\Gamma_{ij}^k)$ of its Christoffel symbols.

It is sufficient to write (1.1) for $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$. This gives

$$(1.2) \quad \partial_k \omega_{ij} = \omega_{il} \Gamma_{kj}^l - \omega_{jl} \Gamma_{ki}^l = \Gamma_{ikj} - \Gamma_{jki}$$

where $\Gamma_{ikj} = \omega_{il} \Gamma_{kj}^l$.

The equality $d\omega = 0$ or $d(\omega_{ij} dx^i \wedge dx^j) = 0$ means

$$(1.3) \quad \partial_k \omega_{ij} + \partial_j \omega_{ki} + \partial_i \omega_{jk} = 0.$$

Recall that ∇ is symmetric (or has no torsion) if

$$(1.4) \quad \nabla_X Y - \nabla_Y X = [X, Y]$$

for arbitrary vector fields X, Y . Taking $X = \partial_i$, $Y = \partial_j$, we can rewrite this in coordinates as $\Gamma_{ij}^k = \Gamma_{ji}^k$ or, equivalently,

$$(1.5) \quad \Gamma_{ijk} = \Gamma_{ikj}.$$

If $d\omega = 0$ then we can locally find Darboux coordinates where ω_{ij} are constants. Then (1.2) is equivalent to the symmetry of Γ_{ijk} with respect to i, k . Therefore in Darboux

coordinates the symmetric connections preserving ω are exactly the connections with the Christoffel symbols Γ_{ijk} which are completely symmetric with respect to all indices i, j, k . Now note that the difference between two connections is a tensor. It follows that for any two symmetric connections ∇, ∇' preserving the same form ω , the difference $\Gamma_{ijk} - \Gamma'_{ijk}$ is a 3-covariant tensor which is completely symmetric in all indices in Darboux coordinates, hence in any coordinates.

Let $\Gamma = (\Gamma_{ij}^k)$ be a linear connection on M . Denote by $\Pi = (\Pi_{ij}^k)$ its symmetric part ($\Pi = \Pi(\Gamma)$) i.e.

$$(1.6) \quad \Pi_{ij}^k = \frac{1}{2}(\Gamma_{ij}^k + \Gamma_{ji}^k).$$

Theorem 1.1. *The map $\Gamma \mapsto \Pi(\Gamma)$ gives a bijective affine correspondence between the set of all connections preserving ω and the set of all symmetric connections. The inverse map $\Pi \mapsto \Gamma$ is given by*

$$(1.7) \quad \Gamma_{kij} = \frac{1}{2}(\partial_k \omega_{ij} - \partial_i \omega_{jk} - \partial_j \omega_{ki}) + (\Pi_{kij} + \Pi_{jik} - \Pi_{ijk}).$$

In case when ω is closed (hence symplectic) this formula can be rewritten as

$$(1.8) \quad \Gamma_{kij} = \partial_k \omega_{ij} + (\Pi_{kij} + \Pi_{jik} - \Pi_{ijk}).$$

Proof. Assume that Γ preserves ω i.e. (1.2) is satisfied. It follows by the cyclic permutation of i, j, k that

$$\partial_i \omega_{jk} = \Gamma_{jik} - \Gamma_{kij}$$

and

$$\partial_j \omega_{ki} = \Gamma_{kji} - \Gamma_{ijk}.$$

Subtracting these two relations from (1.2) and using (1.6) we come to (1.7). Now (1.8) follows if we use (1.3). \square

Corollary 1.2. *If Γ is a connection which preserves ω then the following properties are equivalent:*

- (i) Γ is symmetric (or $\Gamma = \Pi$);
- (ii) Π preserves ω , i.e. the symmetric part of Γ preserves ω .

Proof. Clearly (i) implies (ii). Vice versa, assume that (ii) is true. Then (i) should be true because a connection which preserves ω and has the given symmetric part Π is unique due to the theorem. \square

Corollary 1.3. [Va1] *Let (M, ω) be an almost symplectic manifold with an action of a group G which preserves ω . Assume that there exists a G -invariant connection on M . Then M has a G -invariant connection preserving ω .*

Proof. If Γ is a G -invariant connection on M , then its symmetric part Π is also G -invariant. Therefore the corresponding ω -preserving connection (from Theorem 1.1) will be also G -invariant. \square

Remark 1.4. It is a well known fact that a symmetric connection preserving ω exists if and only if ω is closed (see e.g. [Li1], [To], [Va1]). This can be proved as follows: if Γ is a connection, which preserves ω , i.e. (1.2) is satisfied, then we can rewrite (1.3) in the form

$$\Gamma_{ikj} - \Gamma_{jki} + \Gamma_{kji} - \Gamma_{ijk} + \Gamma_{jik} - \Gamma_{kij} = 0$$

which is obviously true for any symmetric connection.

Vice versa, if ω is closed (hence symplectic), then locally we can take the trivial connection in Darboux coordinates. Globally we can glue a symmetric connection preserving ω using a partition of unity.

Remark 1.5. Suppose we are given non-degenerate symmetric and antisymmetric 2-forms g and ω at the same time. Then we can construct a sequence of connections. Let us start with any antisymmetric torsion tensor T_0 (e.g. zero). Then there exists a unique g -preserving connection Π_1^g with the torsion T_0 . Denote by S_1 its symmetric part (this is a symmetric connection). Then we can find a (unique) ω -preserving connection Γ_1^ω which has the symmetric part S_1 . Denote its torsion tensor by T_1 . Now we can repeat the construction starting with T_1 instead of T_0 . In this way we obtain a sequence of connections and torsion tensors

$$T_0 \mapsto \Pi_1^g \mapsto S_1 \mapsto \Gamma_1^\omega \mapsto T_1 \mapsto \Pi_2^g \mapsto S_2 \mapsto \Gamma_2^\omega \mapsto T_2 \mapsto \dots$$

This sequence looks similar to the sequence which appears when one uses two symplectic structures in the theory of integrable systems.

Note that for Kähler manifolds, starting with $T_0 = 0$ we obtain the sequence which stabilizes i.e. we get $T_j = 0$, $\Pi_j^g = S_j = \Gamma_j^\omega$ for all j .

Remark 1.6. Independently of Theorem 1.1 it is easy to observe a coincidence of functional dimensions (the number of independent functional parameters) of the two sets of connections above: all connections preserving ω and all symmetric connections. Playing with the dimensions we noticed another curious coincidence:

$$(1.9) \quad C_n - C_n^\omega = S_n - S_n^\omega + \Lambda_n^3,$$

where ω is a given non-degenerate (exterior) 2-form,

$$C_n = \text{f-dim of all connections} = n^3,$$

(f-dim means functional dimension)

$$S_n^\omega = \text{f-dim of symmetric connections preserving } \omega = \frac{(n+2)(n+1)n}{6},$$

$$C_n^\omega = \text{f-dim of connections preserving } \omega = \frac{n^2(n+1)}{2},$$

$$S_n = \text{f-dim of all symmetric connections} = \frac{n^2(n+1)}{2},$$

$$\Lambda_n^3 = \text{f-dim of all (external) 3-forms} = \frac{n(n-1)(n-2)}{6}.$$

It would be interesting to clarify the coincidence (1.9) (similarly to the clarification of the “coincidence” $C_n^\omega = S_n$ which was given above).

2. Symplectic connections and their curvature.

Let (M, ω) be a symplectic manifold, ∇ (or Γ) a connection on M .

Definition 2.1. If ∇ is symmetric and preserves the given symplectic form ω then we will say that ∇ is a *symplectic connection*.

Definition 2.2. *Fedosov manifold* is a symplectic manifold with a given symplectic connection. A Fedosov manifold (M, ω, Γ) is called *real-analytic* if M, ω, Γ are all real-analytic.

The curvature tensor of a symplectic connection is defined by the usual formula

$$(2.1) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z.$$

The components of the curvature tensor are introduced by

$$(2.2) \quad R(\partial_j, \partial_k)\partial_i = R^m_{ijk}\partial_m.$$

Denote also

$$(2.3) \quad R_{ijkl} = \omega_{im}R^m_{jkl} = \omega(\partial_i, R(\partial_k, \partial_l)\partial_j).$$

Instead of R_{ijkl} we can also consider $R(X, Y, Z, W)$ which is a multilinear function on any tangent space $T_x M$ (or a function of 4 vector fields which is multilinear over $C^\infty(M)$):

$$(2.3') \quad R(X, Y, Z, W) = \omega(X, R(Z, W)Y),$$

so that $R_{ijkl} = R(\partial_i, \partial_j, \partial_k, \partial_l)$. But it is usually more convenient to do calculations using components.

It is obvious that

$$(2.4) \quad R_{ijkl} = -R_{ijlk}.$$

The formula expressing the components of the curvature tensor in terms of the Christoffel symbols has the standard form:

$$(2.5) \quad R^l_{ijk} = \partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ij} + \Gamma^m_{ki} \Gamma^l_{jm} - \Gamma^m_{ij} \Gamma^l_{km}.$$

For any symmetric connection we have the first Bianchi identity

$$(2.6) \quad R_{i(jkl)} := R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

(For the proofs of (2.5) and (2.6) see e.g. [D-F-N], Sect.30, or [He], Ch.1, Sect.8, 12).

Proposition 2.3. ([Lm2],[Va1]) *For any symplectic connection*

$$(2.7) \quad R_{ijkl} = R_{jikl}.$$

Proof. Let us consider

$$(2.8) \quad \begin{aligned} \partial_k \partial_l \omega_{ij} &= \partial_k [\omega(\nabla_l \partial_i, \partial_j) + \omega(\partial_i, \nabla_l \partial_j)] = \\ &= \omega(\nabla_k \nabla_l \partial_i, \partial_j) + \omega(\nabla_l \partial_i, \nabla_k \partial_j) + \omega(\nabla_k \partial_i, \nabla_l \partial_j) + \omega(\partial_i, \nabla_k \nabla_l \partial_j). \end{aligned}$$

Changing places of k, l and subtracting the result from the (2.8) we obtain:

$$\begin{aligned} 0 &= \omega([\nabla_k, \nabla_l] \partial_i, \partial_j) + \omega(\partial_i, [\nabla_k, \nabla_l] \partial_j) = \omega(R_{ikl}^m \partial_m, \partial_j) + \omega(\partial_i, R_{jkl}^m \partial_m) = \\ &= \omega_{mj} R_{ikl}^m + \omega_{im} R_{jkl}^m = -R_{jikl} + R_{ijkl}. \quad \square \end{aligned}$$

Proposition 2.4. *For any symplectic connection*

$$(2.9) \quad R_{(ijkl)} := R_{ijkl} + R_{lijk} + R_{klij} + R_{jkli} = 0.$$

Proof. Using (2.6) we obtain

$$\begin{aligned} R_{ijkl} + R_{iklj} + R_{iljk} &= 0, \\ R_{jikl} + R_{jkli} + R_{jlki} &= 0, \\ R_{kijl} + R_{kjli} + R_{klij} &= 0, \\ R_{lijk} + R_{ljki} + R_{lkij} &= 0. \end{aligned}$$

Adding all these equalities and using (2.4) and (2.7), we obtain (2.9). \square

3. Ricci tensor and sectional curvature.

Let us look for possible contractions of the curvature tensor which can lead to new non-trivial tensors.

Lemma 3.1. *On any Fedosov manifold*

$$(3.1) \quad R^k_{kij} = \omega^{kl} R_{lkij} = 0.$$

Proof. The result immediately follows from the symmetry of R_{lkij} with respect to the first two subscripts (Proposition 2.3). \square

Definition 3.2. The *Ricci tensor* of a Fedosov manifold is defined by the formula

$$(3.2) \quad K_{ij} = R^k_{ikj} = \omega^{kl} R_{likj} .$$

Equivalently we can define K as a bilinear form on $T_x M$:

$$(3.2') \quad K(X, Y) = K_{ij} X^i Y^j = \text{tr}\{Z \mapsto R(Z, Y)X\} ,$$

where $X, Y, Z \in T_x M$, $X = X^i \partial_i$, $Y = Y^j \partial_j$.

Note that the Ricci tensor depends on connection only. (It does not depend on ω .)

Proposition 3.3. [Va1] *On any Fedosov manifold the Ricci tensor is symmetric:*

$$(3.3) \quad K_{ij} = K_{ji} .$$

Proof. Multiplying (2.9) by ω^{ki} , summing over k, i and using the symmetries (2.4) and (2.7) of the curvature tensor we get the desired result. \square

Corollary 3.4. [Va1] *On any Fedosov manifold*

$$(3.4) \quad \omega^{ji} K_{ij} = 0 .$$

This means that the usual way to define the scalar curvature gives a trivial result.

Now let us show that the second non-trivial contraction of the curvature tensor again leads to the Ricci tensor.

Proposition 3.5. *On any Fedosov manifold*

$$(3.5) \quad \omega^{kl} R_{ijkl} = 2K_{ij} .$$

Proof. Using (2.6) and the symmetries (2.4), (2.7), we obtain

$$\omega^{kl} R_{ijkl} = \omega^{kl} (-R_{iklj} - R_{iljk}) = \omega^{lk} R_{kilj} + \omega^{kl} R_{likj} = 2K_{ij} . \quad \square$$

Remark 3.6. Since the left hand side of (3.5) is symmetric in i, j due to the symmetry (2.7), the formula (3.5) provides another proof of the symmetry of the Ricci tensor.

Yet another definition of the Ricci tensor can be extracted from the following

Proposition 3.7. *Denote*

$$(3.6) \quad L = \omega^{ij} \nabla_i \nabla_j ,$$

which is considered as an operator on vector fields. Then L in fact has order 0 and

$$(3.7) \quad (LX)^i = L^i_j X^j, \quad L^i_j = \omega^{ik} K_{kj},$$

so

$$(3.8) \quad K_{ij} = \omega_{ik} L^k_j.$$

or on other words

$$(3.8') \quad K(X, Y) = \omega(X, LY), \quad X, Y \in T_x M.$$

Proof. We have

$$L = \omega^{ij} \nabla_i \nabla_j = \frac{1}{2} \omega^{ij} (\nabla_i \nabla_j - \nabla_j \nabla_i) = \frac{1}{2} \omega^{ij} R(\partial_i, \partial_j),$$

therefore

$$(LX)^i = \frac{1}{2} \omega^{kl} R^i_{pkl} X^p = \frac{1}{2} \omega^{iq} \omega^{kl} R_{qpkl} X^p = \omega^{iq} K_{qp} X^p,$$

where we used Proposition 3.5. Now (3.7), (3.8) and (3.8') immediately follow. \square

The following definition is a possibility to introduce an analogue of the Einstein equations on a Fedosov manifold.

Definition 3.8. *Einstein equation* on a Fedosov manifold is the system

$$(3.9) \quad K_{ij} = 0.$$

This is a system of nonlinear equations on the data of the Fedosov manifold: its symplectic form and connection. There is an important difference with the symmetric case: the system (3.9) is not determined. Indeed, using Darboux coordinates we can assume that ω has a canonical form and we only seek the Christoffel symbols Γ_{ijk} which should be completely symmetric (see Sect.1). It is easy to see that the functional dimension (the number of free functional parameters) for the space of such connections is $(n+2)(n+1)n/6$ (here $n = \dim_{\mathbb{R}} M$), whereas for the space of symmetric 2-tensors it is $(n+1)n/2$. The equality of these functional dimensions holds if and only if $n = 1$, which can not be a dimension of a Fedosov manifold.

On the other hand the equation (3.9) is non-trivial. Indeed, using (2.5) we obtain in Darboux coordinates

$$(3.10) \quad R_{likj} = \partial_k \Gamma_{lji} - \partial_j \Gamma_{lik} + \omega^{mp} \Gamma_{pji} \Gamma_{lkm} - \omega^{mp} \Gamma_{pik} \Gamma_{ljm},$$

therefore

$$(3.11) \quad K_{ij} = \omega^{kl} R_{likj} = \omega^{kl} \partial_k \Gamma_{lij} - \omega^{kl} \omega^{mp} \Gamma_{pik} \Gamma_{ljm},$$

because after multiplication by ω^{kl} and summation over k, l the second and the third terms in (3.10) vanish due to the symmetry of Γ_{ijk} in i, j, k .

Definition 3.9. *Symplectic Einstein manifold* is a Fedosov manifold such that its Ricci tensor vanishes, i.e. the Einstein equation (3.9) is satisfied.

In terminology of I. Vaisman [Val] such manifolds are called Ricci flat.

Let us discuss a possibility to define sectional curvature. Consider a 2-dimensional subspace Π in a tangent space $T_x M$ to a Fedosov manifold M at a point $x \in M$. For any $X, Y \in \Pi$ consider a quadratic form on Π given by

$$(3.12) \quad E_{X,Y}(Z) = R(Z, Z, Y, X).$$

Assuming that Π is symplectically non-degenerated (non-isotropic) we can choose X, Y , so that $\omega(X, Y) = 1$. Denote by z_1, z_2 the coordinates of $Z \in \Pi$ in the basis X, Y , i.e. $Z = z_1 X + z_2 Y$. Adjusting the basis X, Y by linear transformations preserving ω (i.e. by the action of the group $SL_2(\mathbb{R})$), we can bring $E_{X,Y}$ to one of the following canonical forms:

$$(3.13a) \quad r(z_1^2 + z_2^2), \quad r \in \mathbb{R};$$

$$(3.13b) \quad r(z_1^2 - z_2^2), \quad r > 0;$$

$$(3.13c) \quad \pm z_1^2.$$

Here $r = r(\Pi)$ does not depend on the choice of the basis X, Y in Π and it does not change if we multiply ω by a real constant $c \neq 0$, in particular, if we change sign of ω .

Definition 3.10. *Sectional curvature* of a Fedosov manifold M is a function

$$\Pi \mapsto \{\text{canonical form of } E_{X,Y}\}$$

defined on the set $\text{Gr}_2^\omega(TM)$ of all symplectically non-degenerated (non-isotropic) 2-planes in the tangent bundle TM . We can also interpret it as a function on $\text{Gr}_2^\omega(TM)$ with values in the orbit space $Q(\mathbb{R}^2)/SL_2(\mathbb{R})$ where $Q(\mathbb{R}^2)$ is the linear space of all quadratic forms (or, equivalently, symmetric bilinear forms) on \mathbb{R}^2 , and $SL_2(\mathbb{R})$ acts on $Q(\mathbb{R}^2)$ by the change of variables. In particular, the number $r = r(\Pi)$ from (3.13a, b) becomes a continuous function on $\text{Gr}_2^\omega(TM)$ if we define $r(\Pi) = 0$ in case (3.13c). This function may become singular when we approach the subset of all isotropic planes in the Grassmanian bundle of all 2-planes in TM .

Example 3.11. Consider the 2-sphere S_R^2 of the radius R in \mathbb{R}^3 with the structure of a Fedosov manifold such that the connection is induced by the parallel transport in \mathbb{R}^3 (or, equivalently, is the Levi-Civita connection of the Riemannian metric induced by the canonical metric in \mathbb{R}^3) and the symplectic form is given by the area (induced by the same

Riemannian metric). Then a straightforward calculation shows that for all tangent planes we are in the case (3.13a) with $r(\Pi) = 1/R^2$.

Example 3.12. Consider the standard Lobachevsky (or hyperbolic) plane \mathbb{H}^2 , e.g. the half-plane model $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

As in the previous example the corresponding area-form and Levi-Civita connection provide the structure of Fedosov manifold on \mathbb{H}^2 . A straightforward calculation shows that for all tangent planes we are again in the case (3.13a) but with $r(\Pi) = -1$.

Remark 3.13. Assume that M is a Kähler manifold. Then M has a canonical connection and a canonical curvature map (2.1) (which depends on the connection only), but it has two curvature tensors with all the suffixes down: the Riemannian curvature which we will denote $\tilde{R}_{ijkl} = g(\partial_i, R(\partial_k, \partial_l)\partial_j)$ and the curvature of M as a Fedosov manifold (we will denote it R_{ijkl} as above). There is an obvious relation between them:

$$(3.14) \quad \tilde{R}_{ijkl} = g_{ip}R^p_{jkl} = g_{ip}\omega^{pm}R_{mjkl}.$$

We can write this relation in a different form using the complex structure J from (0.1) (see [Va1]):

$$(3.14') \quad \tilde{R}(X, Y, Z, W) = R(JX, Y, Z, W).$$

Indeed,

$$\tilde{R}(X, Y, Z, W) = g(X, R(Z, W)Y) = \omega(JX, R(Z, W)Y) = R(JX, Y, Z, W).$$

It follows easily from (3.14) (or (3.14')) that on a Kähler manifold only the case (3.13a) is possible. Indeed, given a point $x \in M$ we can find an orthonormal basis in $T_x M$ such that ω has the standard Darboux form in this basis, i.e. $\omega_{2j-1, 2j} = 1$, $\omega_{j, j+p} = 0$ if $p \geq 2$. The first two vectors of this basis can be arbitrary orthonormal vectors $X, Y \in T_x M$ such that $\omega(X, Y) = 1$. Denote by Π the 2-plane spanned by these vectors. Let us calculate the coefficients of the form (3.12) for the chosen basis. We have

$$R_{1121} = \omega_{12}R^2_{121} = \omega_{12}g^{22}\tilde{R}_{2121} = \tilde{R}_{2121} = \tilde{R}_{1212} = \tilde{R}(\Pi),$$

where $\tilde{R}(\Pi)$ is the Riemannian sectional curvature of Π . Similarly we find

$$R_{2221} = \tilde{R}(\Pi), \quad R_{1221} = 0.$$

It follows that in the Kähler case the form (3.12) always belongs to the type (3.13a) with the coefficient $r(\Pi)$ which coincides with the Riemannian sectional curvature.

The following example shows that for general Fedosov manifold the cases (3.13b) and (3.13c) are possible too.

Example 3.14. Let us describe all translation invariant Fedosov structures on \mathbb{R}^2 or on 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. We will use the standard global linear coordinates x^1, x^2 on \mathbb{R}^2 . The Fedosov structure ω, Γ is translation invariant if and only if in these coordinates $\omega_{ij}, \Gamma_{jk}^i$ are constant, or, equivalently, if $\omega_{ij}, \Gamma_{ijk}$ are constant. It follows that Γ_{ijk} should be completely symmetric in i, j, k . Therefore the Fedosov structure in question is completely defined by 5 constants: $\omega_{12}, \Gamma_{111}, \Gamma_{112}, \Gamma_{122}, \Gamma_{222}$.

For simplicity assume that $\omega_{12} = -\omega_{21} = 1$. It follows that $\omega^{21} = -\omega^{12} = 1$. Substituting into (2.5) (or rather into (3.10)), we see that the essential components of the symplectic curvature tensor R_{ijkl} are

$$R_{1112} = 2[(\Gamma_{112})^2 - \Gamma_{111}\Gamma_{122}] ;$$

$$R_{1212} = \Gamma_{112}\Gamma_{122} - \Gamma_{111}\Gamma_{222} ;$$

$$R_{2212} = 2[(\Gamma_{122})^2 - \Gamma_{112}\Gamma_{222}] .$$

Let us take for example $\Gamma_{111} = 0, \Gamma_{112} = a, \Gamma_{122} = 0, \Gamma_{222} = b$. Then

$$R_{1112} = 2a^2, R_{1212} = 0, R_{2212} = -2ab .$$

In particular, the quadratic form (3.12) in the standard basis $X = \partial_1, Y = \partial_2$ is

$$-2a^2 z_1^2 + 2ab z_2^2 .$$

So taking $a > 0, b > 0$ we obtain a translation invariant Fedosov structure (on \mathbb{R}^2 or T^2) with the sectional curvature (3.13b), and taking $a > 0, b = 0$, we obtain a Fedosov structure with the sectional curvature (3.13c).

Calculating the Ricci curvature in this example, we easily find that in the same basis

$$K_{ij} = R_{ij21} ,$$

so the corresponding quadratic form (3.12) coincides with the quadratic form defined by the Ricci tensor.

Note that no Fedosov structure on a compact Riemannian surface M of genus $g \neq 1$ can have sectional curvature (3.13b), (3.13c) at all points, because this would imply existence of a one-dimensional subbundle in TM , hence vanishing of the Euler characteristic of M .

4. Normal coordinates and extensions.

Let us consider a manifold M with a given non-degenerate 2-form ω . Let Γ be a symmetric connection on M . (At the moment we do not assume that Γ preserves ω .) Given a point $p \in M$ we have the exponential map $\exp_p : U \rightarrow M$ defined by Γ . Here U is

a small neighborhood of 0 in $T_p M$, and $\exp_p(v) = x(1)$ where $v \in U$ and $x(t)$ is a geodesic defined in local coordinates (x^1, \dots, x^n) near p by

$$(4.1) \quad \frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad x(0) = p, \quad x'(0) = v,$$

so t is a canonical parameter along the geodesic. Denote by (y^1, \dots, y^n) the local coordinates on M near p which are induced by linear coordinates on $T_p M$ through the map \exp_p , so that

$$(4.2) \quad y^j(p) = 0, \quad \text{and} \quad \frac{\partial y^i}{\partial x_j}(p) = \delta_j^i.$$

In this case (y^1, \dots, y^n) are called *the normal coordinates associated with the local coordinates (x^1, \dots, x^n) at p* .

In other words, if $\xi = y^1 \frac{\partial}{\partial x_1} + \dots + y^n \frac{\partial}{\partial x_n} \in T_p M$ then (y^1, \dots, y^n) are the normal coordinates of $\exp_p \xi$.

Normal coordinates are actually defined by the connection (they do not depend on the form ω). We will list several properties of the normal coordinates (see e.g. [E], [T], [V]).

Let (y^1, \dots, y^n) be local coordinates such that a given point $p \in M$ corresponds to $y^1 = \dots = y^n = 0$. Then they are normal coordinates if and only if

$$(4.3) \quad \Gamma_{jk}^i(y) y^j y^k \equiv 0,$$

where $\Gamma_{jk}^i(y)$ are the Christoffel symbols which are calculated at a point with the local coordinates $y = (y^1, \dots, y^n)$ in the coordinates y^j .

Using ω we can pull the superscript i down and rewrite (4.3) equivalently in the form

$$(4.4) \quad \Gamma_{ijk}(y) y^j y^k \equiv 0.$$

Taking $y^j = t \xi^j$ in (4.4), dividing by t^2 and taking the limit as $t \rightarrow 0$, we obtain due to the symmetry of Γ_{ijk} in j, k :

$$(4.5) \quad \Gamma_{ijk}(0) = 0.$$

Let us write the Taylor expansion of Γ_{ijk} at $y = 0$ in the form:

$$(4.6) \quad \Gamma_{ijk}(y) = \sum_{r=1}^{\infty} \frac{1}{r!} A_{ijk\alpha_1 \dots \alpha_r} y^{\alpha_1} \dots y^{\alpha_r},$$

where

$$(4.7) \quad A_{ijk\alpha_1\dots\alpha_r} = A_{ijk\alpha_1\dots\alpha_r}(p) = \frac{\partial^r \Gamma_{ijk}}{\partial y^{\alpha_1} \dots \partial y^{\alpha_r}} \Big|_{y=0}.$$

Here $p \in M$ is the ‘‘origin’’ of the normal coordinates.

Note that different normal coordinates with the same origin differ by a linear transformation. It follows that the correspondence $p \mapsto A_{ijk\alpha_1\dots\alpha_r}(p)$ defines a tensor on M (the components should be related to our start-up coordinate system (x^1, \dots, x^n)).

Definition 4.1. The tensor $A_{ijk\alpha_1\dots\alpha_r}$ is called an *affine normal tensor* or an *affine extension* of Γ_{ijk} of order $r = 1, 2, \dots$. We also take by definition $A_{ijk} = 0$ which is natural due to (4.5).

Proposition 4.2. *The affine normal tensors have the following properties:*

- (i) $A_{ijk\alpha_1\dots\alpha_r}$ is symmetric in j, k and also in $\alpha_1, \dots, \alpha_r$ (i.e. with respect to the transposition of j and k and also with respect to all permutations of $\alpha_1, \dots, \alpha_r$);
- (ii) for any $r = 1, 2, \dots$

$$(4.8) \quad S_{j,k \leftrightarrow j,k,\alpha_1,\dots,\alpha_r}(A_{ijk\alpha_1\dots\alpha_r}) = 0,$$

where $S_{j,k \leftrightarrow j,k,\alpha_1,\dots,\alpha_r}$ stands for the sum of all the $(r+2)(r+1)/2$ terms obtainable from the one written within parentheses by replacing the pair j, k by any pair from the set $\{j, k, \alpha_1, \dots, \alpha_r\}$.

(iii) Assume that a non-degenerate 2-form ω on M is fixed. If the tensors $\{A_{ijk\alpha_1\dots\alpha_r} \mid r = 1, 2, \dots\}$ are given at one point p , satisfy the conditions in (i), (ii) and besides the series (4.6) is convergent in a neighborhood of 0, then the sums in (4.6) constitute the set of the Christoffel symbols of a symmetric connection in normal coordinates.

The proof easily follows from (4.4) and it does not differ from the proof of the version of this statement for Γ_{jk}^i which is given e.g. in [E],[T],[V]. It can be also reduced to the corresponding statement for Γ_{jk}^i by pulling down the superscript i .

Examples of the identity (4.8) for $r = 1$ and $r = 2$ are as follows:

$$(4.9) \quad A_{ijkl} + A_{ijlk} + A_{iklj} = 0;$$

$$(4.10) \quad A_{ijklm} + A_{ijlkm} + A_{ijmkl} + A_{ikljm} + A_{ikmjl} + A_{ilmjk} = 0.$$

Let us consider an arbitrary tensor $T = (T_{j_1\dots j_l}^{i_1\dots i_k})$ on M .

Definition 4.3. The (affine) *extension* of T of order r is a tensor on M , such that its components at the point $p \in M$ in the local coordinates (x^1, \dots, x^n) are given by the formulas

$$(4.11) \quad T_{j_1\dots j_l,\alpha_1\dots\alpha_r}^{i_1\dots i_k} = \frac{\partial^r T_{j_1\dots j_l}^{i_1\dots i_k}}{\partial y^{\alpha_1} \dots \partial y^{\alpha_r}} \Big|_{y=0},$$

where (y^1, \dots, y^n) are normal coordinates associated with (x^1, \dots, x^n) at p .

In particular we can form the extensions of ω_{ij} which will be tensors $\omega_{ij, \alpha_1 \dots \alpha_r}$.

Note that the first extension of any tensor coincides with its covariant derivative because $\Gamma_{jk}^i(0) = 0$ in normal coordinates.

Let us consider the extensions of ω and Γ together.

Lemma 4.4. (i) *If the connection Γ preserves ω then*

$$(4.12) \quad \omega_{ij, k\alpha_1 \dots \alpha_r} = A_{ikj\alpha_1 \dots \alpha_r} - A_{jki\alpha_1 \dots \alpha_r}$$

for all $r = 0, 1, \dots$ and all $i, j, k, \alpha_1, \dots, \alpha_r$.

(ii) *Vice versa if (4.12) holds on M for $r = 0$ then Γ preserves ω .*

(iii) *If M, ω, Γ are real analytic and the relations (4.12) hold at a point $p \in M$ then Γ preserves ω in the connected component of p .*

Proof. (i) Let us recall (see Sect.1) that Γ preserves ω if and only if

$$(4.13) \quad \partial_k \omega_{ij} = \Gamma_{ikj} - \Gamma_{jki} .$$

Let (y^1, \dots, y^n) be normal coordinates at $p \in M$. Let us write the Taylor expansion of ω_{ij} in these coordinates

$$(4.14) \quad \omega_{ij}(y) = \sum_{r=0}^{\infty} \frac{1}{r!} \omega_{ij, \alpha_1 \dots \alpha_r}(0) y^{\alpha_1} \dots y^{\alpha_r} .$$

It follows that $\partial_k \omega_{ij} = \frac{\partial}{\partial y^k} \omega_{ij}$ has the Taylor expansion

$$\begin{aligned} \partial_k \omega_{ij}(y) &= \sum_{r=1}^{\infty} \frac{1}{r!} \omega_{ij, \alpha_1 \dots \alpha_r}(0) \sum_{s=1}^r y^{\alpha_1} \dots \delta_k^{\alpha_s} \dots y^{\alpha_r} \\ &= \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{s=1}^r \omega_{ij, k\alpha_1 \dots \hat{\alpha}_s \dots \alpha_r}(0) y^{\alpha_1} \dots \hat{y}^{\alpha_s} \dots y^{\alpha_r} . \end{aligned}$$

(Here a “hat” $\hat{}$ over a factor means that this factor should be omitted.) Clearly all the terms in the last sum are equal, therefore we obtain

$$\partial_k \omega_{ij}(y) = \sum_{r=1}^{\infty} \frac{1}{(r-1)!} \omega_{ij, k\alpha_1 \dots \alpha_{r-1}}(0) y^{\alpha_1} \dots y^{\alpha_{r-1}}$$

or

$$(4.15) \quad \partial_k \omega_{ij}(y) = \sum_{r=0}^{\infty} \frac{1}{r!} \omega_{ij, k\alpha_1 \dots \alpha_r}(0) y^{\alpha_1} \dots y^{\alpha_r} .$$

Now comparing the Taylor decompositions in both sides of (4.13) we arrive to (4.12).

(ii) Assuming that (4.12) holds for $r = 0$, we see that the first extension of ω_{ij} vanishes on M , and the statement (ii) follows because this extension coincides with the covariant derivative.

(iii) If M, ω, Γ are real analytic then the series (4.14) converges and can be termwise differentiated which leads to (4.15) (where the series is also convergent). Comparing the Taylor series of the both sides of (4.13) we see that the relations (4.12) at p imply that (4.13) holds in a neighborhood of p which proves (iii). \square

Now we can formulate a point characterization of possible extensions of ω_{ij} and Γ_{ijk} at a point of a Fedosov manifold.

Theorem 4.5. 1) *Assume that we are given a set of tensors*

$$(4.16) \quad \omega_{ij, \alpha_1 \dots \alpha_r} = \omega_{ij, \alpha_1 \dots \alpha_r}(0), \quad A_{ijk\alpha_1 \dots \alpha_r} = A_{ijk\alpha_1 \dots \alpha_r}(0); \quad r = 0, 1, \dots$$

at a point $0 \in \mathbb{R}^n$. Then a structure of a real-analytic Fedosov manifold in a neighborhood of 0 in \mathbb{R}^n with the tensors (4.16) serving as extensions of ω_{ij} and Γ_{ijk} in normal coordinates at 0 , exists if and only if the following conditions are satisfied:

- (a) $A_{ijk}(0) = 0$, and $A_{ijk\alpha_1 \dots \alpha_r}(0)$ is symmetric in j, k as well as in $\alpha_1, \dots, \alpha_r$ for any $r = 1, 2, \dots$;
- (b) $A_{ijk\alpha_1 \dots \alpha_r}(0)$ satisfy the identity (4.8);
- (c) $\omega_{ij}(0)$ is skew-symmetric in i, j and non-degenerate;
- (d) $\omega_{ij, \alpha_1 \dots \alpha_r}(0)$ is skew-symmetric in i, j and symmetric in $\alpha_1, \dots, \alpha_r$;
- (e) for all $r = 0, 1, \dots$ the identity (4.12) is satisfied at 0 ;
- (f) the series (4.6) and (4.14) are convergent for small y .

The structure of a real-analytic Fedosov manifold near 0 is uniquely defined by the tensors (4.16).

2) *Assume that we are only given tensors $A_{ijk\alpha_1 \dots \alpha_r}(0)$ such that the conditions (a), (b) are satisfied and the series (4.6) is convergent for small y . Then a real-analytic symplectic form near 0 complementing the connection (defined by Γ_{ijk} given by the series (4.6)) to a local real-analytic Fedosov manifold structure near 0 exists if and only if for all $r = 1, 2, \dots$*

$$(4.18) \quad A_{ikj\alpha_1 \dots \alpha_r}(0) - A_{jki\alpha_1 \dots \alpha_r}(0) \text{ is symmetric in } k, \alpha_1,$$

i.e. with respect to the transposition of k and α_1 . The form ω above is uniquely defined by $\omega_{ij}(0)$ and it exists for arbitrary skew-symmetric non-degenerate $\omega_{ij}(0)$.

Proof. 1) It is obvious from the considerations above that the conditions (a)-(f) are necessary for the existence of a local real-analytic Fedosov manifold near 0 with the given extensions (4.16). They are also sufficient because we can define $\omega_{ij}(y)$ and $\Gamma_{ijk}(y)$ near 0 by the convergent series (4.6) and (4.14), and it is easy to see that all the conditions will be satisfied. The uniqueness of ω, Γ near 0 is obvious because the tensors (4.16) define all the Taylor coefficients of ω, Γ at 0 .

2) If the tensors $A_{ijk\alpha_1\dots\alpha_r}$ satisfy (a),(b) and the series (4.6) are convergent for small y , then the sums of these series provide $\Gamma_{ijk}(y)$ which determine a real-analytic symmetric connection in normal coordinates for any given real-analytic non-degenerate skew-symmetric form $\omega_{ij}(y)$ defined near 0. To make this 2-form symplectic and preserved by Γ we can try to find it from the conditions (4.13) which determine all first derivatives $\partial_k\omega_{ij}$ near 0, so we only need to know $\omega_{ij}(0)$.

Let us choose an arbitrary skew-symmetric matrix $\omega_{ij}(0)$. Then we can find $\omega_{ij}(y)$ from (4.13) if and only if the following compatibility conditions are satisfied near 0:

$$(4.19) \quad \partial_l(\Gamma_{ikj} - \Gamma_{jki}) = \partial_k(\Gamma_{ilj} - \Gamma_{jli}) ,$$

or, equivalently,

$$(4.20) \quad \partial_l(\Gamma_{ikj} - \Gamma_{jki}) \text{ is symmetric in } k, l .$$

But the same argument as the one leading to (4.15), shows that the Taylor expansion of $\partial_l\Gamma_{ikj}(y)$ at 0 has the form

$$(4.21) \quad \partial_l\Gamma_{ikj}(y) = \sum_{r=1}^{\infty} \frac{1}{r!} A_{ikjl\alpha_1\dots\alpha_r} y^{\alpha_1} \dots y^{\alpha_r} .$$

It follows that (4.20) is equivalent to (4.18), so the statement 2) follows. \square

Corollary 4.6. *Let $A_{ijk\alpha_1\dots\alpha_r}$ be tensors at a point $0 \in \mathbb{R}^n$, $r = 0, 1, \dots$, $A_{ijk} = 0$ and the series (4.6) are convergent near 0. Then the sum of these series define symplectic connection in normal coordinates, which are at the same time Darboux coordinates (with arbitrary constant skew-symmetric non-degenerate ω_{ij}), if and only if $A_{ijk\alpha_1\dots\alpha_r}$ are symmetric in i, j, k and in $\alpha_1, \dots, \alpha_r$, and besides the conditions (4.8) are satisfied.*

Proof. Let us fix an arbitrary constant skew-symmetric non-degenerate ω_{ij} . The condition (4.8) guarantees that the sums $\Gamma_{ijk}(y)$ of the series (4.6) give us components of a symmetric connection in normal coordinates due to Proposition 4.2. Now (1.2) implies that this connection preserves the constant form ω if and only if Γ_{ijk} is symmetric in ijk which is equivalent to saying that $A_{ijk\alpha_1\dots\alpha_r}$ are symmetric in i, j, k for any $r = 1, 2, \dots$. \square

Now we can formulate the simplest result about local invariants of Fedosov manifolds. For simplicity we will only consider invariants which are tensors though similar results are true e.g. for invariants which are tensors with values in density bundles.

Definition 4.7. A tensor $T = (T_{j_1\dots j_l}^{i_1\dots i_k}(x))$ on a Fedosov manifold M is called a *local invariant* of M if its components at each point $p \in M$ are functions of a finite number of derivatives (of arbitrary order, including 0) of Γ_{ijk} and ω_{ij} taken at the same point p in local coordinates, i.e.

$$(4.22) \quad T_{j_1\dots j_l}^{i_1\dots i_k}(p) = F_{j_1\dots j_l}^{i_1\dots i_k}(\Gamma_{ijk}(p), \partial_\alpha\Gamma_{ijk}(p), \dots, \omega_{ij}(p), \partial_\alpha\omega_{ij}(p), \partial_{\alpha_2}\partial_{\alpha_1}\omega_{ij}(p), \dots) ,$$

where the functions $F_{j_1 \dots j_l}^{i_1 \dots i_k}$ do not depend on the choice of the local coordinates. A local invariant is called *rational* or *polynomial* if the functions $F_{j_1 \dots j_l}^{i_1 \dots i_k}$ are all rational or polynomial respectively.

In particular, this definition applies to local invariants which are differential forms.

Theorem 4.8. *Any local invariant of a Fedosov manifold is a function of ω_{ij} and a finite number of its affine normal tensors, i.e. it can be presented in the form*

$$(4.23) \quad T_{j_1 \dots j_l}^{i_1 \dots i_k}(p) = G_{j_1 \dots j_l}^{i_1 \dots i_k}(\omega_{ij}(p), A_{ijk\alpha_1}(p), \dots, A_{ijk\alpha_1 \dots \alpha_r}(p)) ,$$

where the functions $G_{j_1 \dots j_l}^{i_1 \dots i_k}$ do not depend on the choice of local coordinates. If the invariant $T_{j_1 \dots j_l}^{i_1 \dots i_k}$ is rational or polynomial, then the functions $G_{j_1 \dots j_l}^{i_1 \dots i_k}$ are also rational or polynomial respectively.

Proof. Since T is a tensor, its components $T_{j_1 \dots j_l}^{i_1 \dots i_k}(p)$ will not change if we replace local coordinates (x^1, \dots, x^n) by normal coordinates (y^1, \dots, y^n) which are associated with (x^1, \dots, x^n) at p . But then according to Definition 4.7 we obtain, using formulas (4.7) and (4.12):

$$(4.24) \quad T_{j_1 \dots j_l}^{i_1 \dots i_k}(p) = F_{j_1 \dots j_l}^{i_1 \dots i_k}(0, A_{ijk\alpha}(p), \dots, \omega_{ij}(p), 0, A_{i\alpha_1 j \alpha_2}(p) - A_{j\alpha_1 i \alpha_2}(p), \dots) .$$

All statements of the Theorem immediately follow. \square

5. Normal tensors and curvature.

A. Let us try to connect the affine normal tensors and extensions of ω_{ij} on a Fedosov manifold M with the curvature tensor. First let us rewrite (2.5) in terms of Γ_{ijk} and R_{ijkl} instead of Γ_{jk}^i and R_{jkl}^i :

$$(5.1) \quad R_{ijkl} = \omega_{is} \partial_k (\omega^{sp} \Gamma_{pjl}) - \omega_{is} \partial_l (\omega^{sp} \Gamma_{pjk}) - \omega^{mp} \Gamma_{pjl} \Gamma_{ikm} + \omega^{mp} \Gamma_{pjk} \Gamma_{ilm} .$$

The standard formula for differentiation of the inverse matrix gives

$$(5.2) \quad \partial_k \omega^{ij} = -\omega^{ir} \omega^{sj} \partial_k \omega_{rs} = \omega^{ir} \omega^{js} \partial_k \omega_{rs} .$$

Let us introduce normal coordinates with the center p . Observe that in such coordinates $\Gamma_{ijk}(p) = (\partial_k \omega_{ij})(p) = 0$. It follows from (5.2) that $(\partial_k \omega^{ij})(p) = 0$. Now performing differentiation in the right hand side of (5.1) and taking the values at p we obtain

$$(5.3) \quad R_{ijkl} = A_{ijlk} - A_{ijkl} ,$$

which is an equality of tensors, and it is true everywhere on M because $p \in M$ is an arbitrary point. This gives an expression of R_{ijkl} as a linear combination of affine normal tensors.

We can do another step in this direction by applying ∂_m to both sides of (5.1) and taking values at p which leads to

$$(5.4) \quad R_{ijkl,m} = A_{ijlkm} - A_{ijklm} ,$$

where $R_{ijkl,m}$ means the first extension of the curvature tensor R_{ijkl} or its first covariant derivative (which coincide).

Further differentiations lead to expressions of the extensions of the curvature tensor which are already non-linear in normal tensors and also explicitly contain ω^{ij} , though they are linear in the highest order normal tensors:

Lemma 5.1. *For any $r \geq 0$*

$$(5.5) \quad R_{ijkl,\alpha_1 \dots \alpha_r} = A_{ijlk\alpha_1 \dots \alpha_r} - A_{ijkl\alpha_1 \dots \alpha_r} + P_r$$

where P_r is a polynomial of the components ω^{ij} and of the normal tensors of order $\leq r$, $P_0 = P_1 = 0$.

B. Our goal now is to express the normal tensors and the extensions of ω_{ij} in terms of the curvature tensor. Let us start by solving the equations (5.3) with respect to A_{ijkl} . Making cyclic permutations of j, k, l we obtain, using (4.9):

$$(5.3') \quad R_{iklj} = A_{ikjl} - A_{iklj} = A_{ijkl} - A_{iklj} = 2A_{ijkl} + A_{ijlk} ,$$

$$(5.3'') \quad R_{iljk} = A_{ilkj} - A_{iljk} = A_{iklj} - A_{ijlk} = -2A_{ijlk} - A_{ijkl} .$$

Adding doubled (5.3') with (5.3'') gives

$$(5.6) \quad A_{ijkl} = \frac{1}{3}(2R_{iklj} + R_{iljk}) .$$

Also using (2.6), we obtain

$$(5.7) \quad A_{ijkl} = \frac{1}{3}(R_{iklj} + R_{ijlk}) .$$

A simple relation between the second extension of ω_{ij} and the curvature tensor is given by

Proposition 5.2. *On any Fedosov manifold*

$$(5.8) \quad \omega_{ij,kl} = \frac{1}{3}R_{klij} .$$

Proof. According to Lemma 4.4 we obtain, using (5.7):

$$\omega_{ij,kl} = A_{ikjl} - A_{jkil} = \frac{1}{3}(R_{iklj} + R_{ijlk} - R_{jkli} - R_{jilk}) .$$

The second and fourth terms cancel due to Proposition 2.3, so using (2.6) we obtain:

$$\omega_{ij,kl} = \frac{1}{3}(R_{iklj} - R_{jkli}) = \frac{1}{3}(R_{kilj} - R_{kjli}) = \frac{1}{3}(-R_{kijl} - R_{kjl i}) = \frac{1}{3}R_{kl ij} . \quad \square$$

Now we will try to solve the equations (5.5) with respect to the higher order normal tensors. Relations of this type between the curvature tensor R^i_{jkl} and normal tensors $A^i_{jk\alpha_1\dots\alpha_r}$ defined similarly to (4.7) but with Γ_{ijk} replaced by Γ^i_{jk} , can be found in [Th], Sect.49, and [V], Ch. VI, Sect.9-12. Note that generally $A^i_{jk\alpha_1\dots\alpha_r}$ does not coincide with $\omega^{ip}A_{pj k\alpha_1\dots\alpha_r}$ because ω^{ip} is not constant. We will give more details than in the exposition in [Th] and [V] which we found insufficient.

We will need a combinatorial preparation first. Let $J = \{j_1 j_2 \dots, j_m\}$ be an ordered set of positive integers. We associate to J an ordered set $T(J)$ of ordered triads of elements of J . The set $T(J)$ will be defined by an inductive procedure:

$$i) T(J) = \emptyset, \quad m = 1, 2;$$

$$ii) T(J) = \{(j_1 j_2 j_3) < (j_3 j_1 j_2)\}, \quad m = 3;$$

iii) For $m > 3$ the elements $T(J)$ in the increasing order are:

$$\begin{aligned} & (j_1 j_2 j_3), (j_1 j_3 j_4), \dots, (j_1 j_{m-1} j_m), \\ & (j_m j_1 j_2), \\ & (j_2 j_m j_{m-1}), (j_2 j_{m-1} j_{m-2}), \dots, (j_2 j_4 j_3), \\ & (j_3 j_2 j_4), \\ & T(J \setminus \{j_1 j_2\}). \end{aligned}$$

Remark 5.3. The most important property of this ordering is the following: the first and third elements of any triad are the same pair (with possibly reversed order) as the first two elements of the next triad.

Lemma 5.4. For $m \geq 2$ we have

$$(5.9) \quad \text{Card}(T(J)) = m(m-1)/2 - 1 .$$

Proof. The formula holds for $m = 2$ and $m = 3$. Now let us use induction with respect to m . Denote $N_m = \text{Card}(T(M))$. By the construction

$$N_m = (m-2) + 1 + (m-3) + 1 + N_{m-2} = N_{m-2} + (2m-3).$$

On the other hand it is easy to check that

$$m(m-1)/2 = (m-2)(m-3)/2 + (2m-3). \quad \square$$

Let $R_{ijkl, \alpha_1 \dots \alpha_r}$ be the r -th extension of the curvature tensor R_{ijkl} on a Fedosov manifold. Consider the ordered set $J = \{jkl\alpha_1 \dots \alpha_r\}$. Note that we ignore the first subscript i which plays a special role. For any triad $u = (u_1 u_2 u_3) \in T(J)$ consider the r -th extension \tilde{R}_{iu} of the curvature tensor R_{iu} :

$$\tilde{R}_{iu} = R_{iu, J \setminus u}.$$

Denote $N = \text{Card}(T(J))$ and let $u^{(1)} < u^{(2)} < \dots < u^{(N)}$ be the list of all elements of $T(J)$ (in increasing order).

The following theorem is an analogue of a Veblen theorem from [V].

Theorem 5.5. *For any $r \geq 0$ the affine normal tensor of order $r+1$ can be expressed through the extensions of the curvature tensor as follows:*

$$(5.10) \quad A_{ijkl\alpha_1 \dots \alpha_r} = -\frac{1}{N+1} \sum_{s=1}^N (N-s+1) \tilde{R}_{iu^{(s)}} + P_r,$$

where P_r is a polynomial of ω^{ij} and affine normal tensors of order $p \leq r$. Besides $P_0 = P_1 = 0$, so for $r = 0, 1$ we have precise formulas

$$(5.10)_0 \quad A_{ijkl} = -\frac{1}{3}(2R_{ijkl} + R_{iljk});$$

$$(5.10)_1 \quad A_{ijklm} = -\frac{1}{6}(5R_{ijkl,m} + 4R_{ijlm,k} + 3R_{imjk,l} + 2R_{ikml,j} + R_{ilkm,j}).$$

To prove the theorem we will need several simple facts. We recall, first of all, that the normal tensors $A_{ijk\alpha_1, \dots, \alpha_r}$ are symmetric in j, k and also in $\alpha_1, \dots, \alpha_r$.

Let us fix the ordered set of indices $J = \{j, k, \alpha_1, \dots, \alpha_r\}$. Denote by $P(J)$ the set of all ordered pairs $p = (p_1, p_2)$ of elements from J (here “ordered” means that p_1 stands somewhere before p_2 in the list of the elements of J above). Denote by \tilde{A}_{ip} the normal tensor $A_{ip_1 p_2, J \setminus p}$ of degree r . The equality (4.8) (in Proposition 4.2) can be rewritten as follows:

$$(5.11) \quad \sum_{p \in P(J)} \tilde{A}_{ip} = 0.$$

Lemma 5.6. (i) *The extension of order $r \geq 1$ of the tensor ω_{ij} is a linear combination of normal tensors of order $r-1$.*

(ii) *Every component of the extension of order $r \geq 1$ of the tensor ω^{ij} is a polynomial (independent on the choice of the coordinates) of the components ω^{ij} and normal tensors of order $\leq r-1$.*

Proof. The part (i) is obvious from Lemma 4.4 (in fact the formula (4.12) gives an explicit expression). The part (ii) is obtained by multiple differentiation of the formula (5.2) in normal coordinates with substitution of the right hand side of (5.2) instead of the first derivatives of ω^{ij} after each differentiation, and further use of the part (i) (or formula (4.12)). \square

Proof of Theorem 5.5. Let $J = \{j, k, l, \alpha_1, \dots, \alpha_r\}$. Applying Lemma 5.1 and using the symmetries of normal tensors one could write that

$$\begin{aligned} & - \sum_{s=1}^N (N-s+1) \tilde{R}_{iu^{(s)}} = N \tilde{A}_{ijk} - \\ & \quad - (N \tilde{A}_{ijl} - (N-1) \tilde{A}_{ijl}) - \\ & - ((N-1) \tilde{A}_{ij\alpha_1} - (N-2) \tilde{A}_{ij\alpha_1}) - \dots = \\ & = N \tilde{A}_{ijk} - \sum_{p \in P(J), p \neq (jk)} \tilde{A}_{ip} \end{aligned}$$

modulo a polynomial of ω^{ij} and normal tensors of order $\leq r$ (this polynomial vanishes if $r = 0$ or 1). Now the theorem follows because the right hand side equals $(N+1) \tilde{A}_{ijk}$ due to (5.11). \square

Corollary 5.7. *Any component $A_{ijk\alpha_1 \dots \alpha_r}$ of the normal tensor of order r can be expressed as a universal polynomial (i.e. a polynomial independent of the coordinates) of components ω^{ij} and components of extensions of the curvature tensor R_{ijkl} of order $\leq r-1$.*

Proof. The result follows from Theorem 5.5 if we use (5.6) (or (5.7)) and then implement induction in r . \square

C. Now we will establish relations between extensions and covariant derivatives. Let us consider an arbitrary tensor $T = (T_{j_1 \dots j_l}^{i_1 \dots i_k})$. Let us recall that its (first order) covariant derivative is a tensor $\nabla_{\partial_\alpha} T_{j_1 \dots j_l}^{i_1 \dots i_k} = \nabla_\alpha T_{j_1 \dots j_l}^{i_1 \dots i_k} = T_{j_1 \dots j_l, \alpha}^{i_1 \dots i_k}$ defined by the formulas

$$\begin{aligned} (5.12) \quad & T_{j_1 \dots j_l, \alpha}^{i_1 \dots i_k} = \nabla_\alpha T_{j_1 \dots j_l}^{i_1 \dots i_k} = \nabla_{\partial_\alpha} T_{j_1 \dots j_l}^{i_1 \dots i_k} = \\ & \partial_\alpha T_{j_1 \dots j_l}^{i_1 \dots i_k} + \Gamma_{\alpha s}^{i_1} T_{j_1 \dots j_l}^{s i_2 \dots i_k} + \dots + \Gamma_{\alpha s}^{i_k} T_{j_1 \dots j_l}^{i_1 \dots i_{k-1} s} - \Gamma_{\alpha j_1}^s T_{s j_2 \dots j_l}^{i_1 \dots i_k} - \dots - \Gamma_{\alpha j_l}^s T_{j_1 \dots j_{l-1} s}^{i_1 \dots i_k} = \\ & \partial_\alpha T_{j_1 \dots j_l}^{i_1 \dots i_k} + \omega^{i_1 q} \Gamma_{q \alpha s} T_{j_1 \dots j_l}^{s i_2 \dots i_k} + \dots + \omega^{i_k q} \Gamma_{q \alpha s} T_{j_1 \dots j_l}^{i_1 \dots i_{k-1} s} - \\ & \quad \omega^{s q} \Gamma_{q \alpha j_1} T_{s j_2 \dots j_l}^{i_1 \dots i_k} - \dots - \omega^{s q} \Gamma_{q \alpha j_l} T_{j_1 \dots j_{l-1} s}^{i_1 \dots i_k} . \end{aligned}$$

In particular, taking normal coordinates we immediately see that the covariant derivative of T coincides with the first extension of T .

We define the higher order covariant derivative of T of order r as the following tensor:

$$(5.13) \quad T_{j_1 \dots j_l, \alpha_1, \alpha_2, \dots, \alpha_r}^{i_1 \dots i_k} = \nabla_{\alpha_r} \dots \nabla_{\alpha_1} T_{j_1 \dots j_l}^{i_1 \dots i_k} .$$

It depends on the order of $\alpha_1, \dots, \alpha_r$ (unlike the extension $T_{j_1 \dots j_l, \alpha_1 \dots \alpha_r}^{i_1 \dots i_k}$).

Proposition 5.8. (i) *Every component of the covariant derivative of T of order r can be expressed through extensions of T of order $\leq r$:*

$$(5.14) \quad T_{j_1 \dots j_l, \alpha_1, \alpha_2, \dots, \alpha_r}^{i_1 \dots i_k} = T_{j_1 \dots j_l, \alpha_1 \alpha_2 \dots \alpha_r}^{i_1 \dots i_k} + P_{r-1} ,$$

where P_{r-1} is a linear combination of components of extensions of T of order $\leq r-1$ with coefficients which are universal polynomials of ω^{ij} and of the components of the normal tensors of order $\leq r-1$.

(ii) *Every component of the extension of T of order r can be expressed through the covariant derivatives of T of order $\leq r$:*

$$(5.15) \quad T_{j_1 \dots j_l, \alpha_1 \alpha_2 \dots \alpha_r}^{i_1 \dots i_k} = T_{j_1 \dots j_l, \alpha_1, \alpha_2, \dots, \alpha_r}^{i_1 \dots i_k} + P_{r-1} ,$$

where P_{r-1} is a linear combination of components of covariant derivatives of T of order $\leq r-1$ with coefficients which are universal polynomials of ω^{ij} and of the components of the normal tensors of order $\leq r-1$.

Proof. (i) is obtained by repeated application of covariant differentiation formula (5.12) in normal coordinates with the use of (5.2) to differentiate ω^{ij} . We should also use (1.2) to get rid of the first derivatives of ω_{ij} as soon as they show up, replacing them by linear combinations of Γ_{ijk} . After all differentiations are done, we should take values of all quantities at the center of the normal coordinates, which leads to (5.14).

To prove (ii) we should use induction in r . We already observed that (5.15) is true for $r=1$ (with $P_0=0$), i.e. the first extension coincides with the first covariant derivative. Now assuming that (5.15) is true with $r \leq r_0-1$, we can take (5.14) with $r=r_0$ and replace components of the extensions of T in P_{r-1} by the corresponding combinations of the components of the covariant derivatives. This leads to (5.15) with $r=r_0$. \square

Theorem 5.9. *Any component $A_{ijk\alpha_1 \dots \alpha_r}$ of the normal tensor of order r can be expressed as a universal polynomial (i.e. a polynomial independent of the coordinates) of the components ω^{ij} and of the components of covariant derivatives of the curvature tensor R_{ijkl} of order $\leq r-1$.*

Proof. Let us act by induction in r . For $r=1$ the statement follows from (5.6) (or (5.7)). For any $r \geq 2$ let us use Theorem 5.5 to express any component of the normal tensor of order $r+1$ through extensions of the curvature tensor of order $\leq r$ by the formula (5.10). In the remainder term P_r of (5.10) we should replace the normal tensors of order $\leq r$ by the covariant derivatives of the curvature tensor of order $\leq r-1$ using the induction assumption. Now express the extensions $\tilde{R}_{iu(s)}$ in the right hand side of (5.10) by the covariant derivatives of the curvature tensor using Proposition 5.8. It remains to use again the induction assumption to replace the newly appeared components of the normal tensors (from the remainders in (5.15)) by the components of the covariant derivatives of R_{ijkl} . \square

Corollary 5.10. *Proposition 5.8. remains true if in its statement we replace the components of the normal tensors of order $\leq r-1$ by the components of the covariant derivatives of order $\leq r-2$ of the curvature tensor R_{ijkl} .*

Now we are ready for another presentation of possible local invariants of a Fedosov manifold.

Theorem 5.11. *Any local invariant of a Fedosov manifold is a function of ω_{ij} and a finite number of covariant derivatives of its curvature tensor R_{ijkl} , i.e. it can be presented in the form*

$$(5.16) \quad T_{j_1 \dots j_l}^{i_1 \dots i_k}(p) = G_{j_1 \dots j_l}^{i_1 \dots i_k}(\omega_{ij}(p), R_{ijkl}(p), \dots, R_{ijkl, \alpha_1, \dots, \alpha_r}(p)),$$

where the functions $G_{j_1 \dots j_l}^{i_1 \dots i_k}$ do not depend on the choice of local coordinates. If the invariant $T_{j_1 \dots j_l}^{i_1 \dots i_k}$ is rational or polynomial (in $\omega_{ij}, \omega^{ij}, \Gamma_{ijk}$ and derivatives of $\omega_{ij}, \Gamma_{ijk}$), then the functions $G_{j_1 \dots j_l}^{i_1 \dots i_k}$ are also rational or polynomial (in the same sense) respectively.

Proof. Due to Theorem 4.8 and Proposition 5.2 it is sufficient to express the normal tensors $A_{ijk\alpha_1 \dots \alpha_r}$ polynomially in terms of ω^{ij} and the covariant derivatives $R_{ijkl, \alpha_1, \dots, \alpha_r}$, $p = 0, \dots, r-1$. This is possible due to Theorem 5.9. \square

Remark 5.12. Not all functions (and even not all polynomials) $G_{j_1 \dots j_l}^{i_1 \dots i_k}$ indeed give us local invariants of the Fedosov manifold (the corresponding expression is not necessarily a tensor yet, it might still depend on the choice of local coordinates). The form given in the expression (5.16) is necessary but not sufficient for an expression to be a local invariant.

D. Theorem 4.5 establishes necessary and sufficient conditions for the tensors $A_{ijk\alpha_1 \dots \alpha_r}$, given at a point, to be normal tensors of a real-analytic Fedosov manifold. The relations between normal tensors and the extensions of the curvature tensor or with its covariant derivatives which we proved above, in principle allow to write necessary and sufficient conditions for a family of tensors to be a family of extensions or covariant derivatives of the curvature tensor of a real-analytic Fedosov manifold at a point. However the algebraic relations between normal tensors and the curvature are not simple, so we can do it effectively on the lowest levels only. As an example we will do it for the curvature tensor itself (without extensions or derivatives) and also for the first covariant derivatives of the curvature tensor.

Theorem 5.13. *Assume that we are given a tensor R_{ijkl} at the origin $0 \in \mathbb{R}^n$. Then necessary and sufficient conditions for this tensor to be the curvature tensor of a Fedosov manifold at a point are:*

$$(a) \quad R_{ijkl} = -R_{ijlk} ;$$

$$(b) \quad R_{ijkl} = R_{jikl} ;$$

$$(c) \quad R_{i(jkl)} := R_{ijkl} + R_{iklj} + R_{iljk} = 0 .$$

Proof. The conditions (a), (b) (c) are necessary as was established in Sect. 2. Let us prove that they are sufficient.

Let us introduce a hypothetical normal tensor A_{ijkl} at 0 by the formulas (5.6) or (5.7) (it follows from (a) and (c) that they are equivalent). Let us check that the tensor A_{ijkl} satisfies the conditions from the part 2) of Theorem 4.5 with $r = 1$ i.e.

$$(5.17) \quad A_{ijkl} = A_{ikjl} ;$$

$$(5.18) \quad A_{i(jkl)} = 0 ;$$

$$(5.19) \quad A_{ijkl} - A_{kjil} - A_{ilkj} + A_{klij} = 0 .$$

Clearly (5.17) follows from (5.7), and (5.18) follows from the condition (c). Now substituting the expressions for the terms in the left hand side of (5.19) from (5.7) we get

$$\frac{1}{3}(R_{iklj} + R_{ijlk} - R_{kilj} - R_{kjli} - R_{ikjl} - R_{iljk} + R_{kijl} + R_{klji}) = -\frac{1}{3}(R_{ijkl} + R_{jkli} + R_{klij} + R_{lijk}),$$

where we used the properties (a) and (b). The last expression vanishes due to Proposition 2.4 (it is established in its proof that (2.9) follows from the properties (a), (b), (c)).

Now defining $A_{ijk\alpha_1\alpha_2\dots\alpha_r} = 0$ for all $r \geq 2$ we obtain a system of tensors which can serve as a system of normal tensors of a Fedosov manifold at a point due to Theorem 4.5. It remains to prove that its curvature tensor \tilde{R}_{ijkl} at the given point will coincide with the given tensor R_{ijkl} . By (5.2) we have

$$\tilde{R}_{ijkl} = A_{ijlk} - A_{ijkl} = \frac{1}{3}(2R_{ilkj} + R_{ikjl} - 2R_{iklj} - R_{iljk}) = \frac{1}{3}(3R_{ilkj} + 3R_{ikjl}) = R_{ijkl} ,$$

where we used the properties (a) and (c). \square

To work with the first derivatives we will need another expression of the normal tensors A_{ijklm} through the first covariant derivatives (or first extensions) of R_{ijkl} . It is similar to (5.10)₁ but has smaller coefficients.

Lemma 5.14. *On any Fedosov manifold*

$$(5.20) \quad A_{ijklm} = -\frac{1}{6}(2R_{ijkl,m} + R_{ijkm,l} + R_{ikjl,m} + R_{ikjm,l} + R_{iljm,k}).$$

Proof. The formula (5.20) can be deduced from a similar formula (49.7) in [Th], p.130, by pulling down the superscript. However for the convenience of the reader we will supply the proof which is only sketched in [Th].

The idea of the proof is to eliminate all the terms in (4.10), except the first one, using (5.4) so that the corresponding curvature terms are added. As a first step substituting $A_{ijlkm} = A_{ijklm} + R_{ijkl,m}$ to (4.10) we obtain

$$(5.21) \quad 2A_{ijklm} + R_{ijkl,m} + A_{ijmkl} + A_{ikljm} + A_{ikmj l} + A_{ilmjk} = 0 .$$

Now using the symmetry relations in Proposition 4.2(i), we can write

$$(5.22) \quad A_{ijmkl} = R_{ijkm,l} + A_{ijkml} = R_{ijkm,l} + A_{ijkml} ,$$

so (5.21) becomes

$$(5.23) \quad 3A_{ijklm} + R_{ijkl,m} + R_{ijkm,l} + A_{ikljm} + A_{ikmj,l} + A_{ilmjk} .$$

In the same way we can treat all components of the normal tensors in (5.21) such that either j or k (but not both) occurs as the the second or third subscript. Eliminating in this way A_{ikljm} and $A_{ikmj,l}$ from (5.23) we obtain

$$(5.24) \quad 5A_{ijklm} + R_{ijkl,m} + R_{ijkm,l} + R_{ikjl,m} + R_{ikjm,l} + A_{ilmjk} = 0 .$$

To eliminate the last term we should do the same operation twice to put both j and k to the second and third place:

$$A_{ilmjk} = R_{iljm,k} + A_{iljmk} = R_{iljm,k} + A_{iljmk} = R_{iljm,k} + R_{ijkl,m} + A_{ijklm} .$$

Substituting this expression for A_{ilmjk} into (5.24) we arrive to (5.20). \square

Proposition 5.15. (The second Bianchi identity) *On any Fedosov manifold*

$$(5.25) \quad R_{ij(kl,m)} := R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0 .$$

Proof. This identity is again true in a much more general context. In particular, we can deduce it from the similar identity for arbitrary symmetric connections (with the subscript i lifted – see e.g. (49.13) in [Th], p.132). But for the convenience of the reader we will reproduce the proof from [Th]. Another proof can be obtained by use of normal coordinates as in [ON].

Let us interchange l and m in (5.20) and subtract the result from (5.20) using the identities obtained from the identities (a), (b), (c) from Theorem 5.13 by taking the first covariant derivative. We will get then:

$$\begin{aligned} 0 &= 2R_{ijkl,m} - 2R_{ijkm,l} + R_{ijkm,l} - R_{ijkl,m} + R_{ikjl,m} - R_{ikjm,l} + \\ &\quad + R_{ikjm,l} - R_{ikjl,m} + R_{iljm,k} - R_{imjl,k} = \\ &\quad R_{ijkl,m} - R_{ijkm,l} + R_{iljm,k} - R_{imjl,k} = \\ &\quad R_{ijkl,m} + R_{ijmk,l} + (-R_{ilmj,k} - R_{imjl,k}) = \\ &\quad R_{ijkl,m} + R_{ijmk,l} + R_{ijlm,k} . \quad \square \end{aligned}$$

Proposition 5.16. (Integrability identity.) *On any Fedosov manifold*

$$(5.26) \quad R_{imkj,l} + R_{ijml,k} + R_{iljk,m} + R_{iklm,j} = 0 .$$

Proof. Let us use the identity (4.18) with $r = 2$ i.e.

$$(5.27) \quad A_{ijklm} - A_{kjilm} - A_{ilkjm} + A_{klijm} = 0 .$$

Substituting here the expressions of A_{ijklm} and other terms through the components $R_{ijkl,m}$ given by Lemma 5.14, and using (2.4), (2.7) we obtain:

$$\begin{aligned} -6(A_{ijklm} - A_{kjilm} - A_{ilkjm} + A_{klijm}) &= 2(R_{ijkl,m} + R_{jkli,m} + R_{klij,m} + R_{lijk,m}) + \\ &+ (R_{jikm,l} + R_{jkmi,l}) + (R_{iljm,k} + R_{ijml,k}) + (R_{limk,j} + R_{lkim,j}) + (R_{klmj,i} + R_{kjlmi,i}) . \end{aligned}$$

The expression in the first parentheses in the right hand side vanishes because the differentiated identity (2.9) gives $R_{(ijkl),m} = 0$. Now using the differentiated first Bianchi identity (2.6) we can rewrite the right hand side as

$$-(R_{milj,k} + R_{mjik,l} + R_{mkjl,i} + R_{mlki,j}) .$$

This expression should vanish due to (5.27). Interchanging m and i , as well as k and l , we arrive to (5.26). \square

Remark 5.17. The combinatorial structure of the integrability identity (5.25) has the following interesting features.

1) The first subscript i is fixed, but each of other subscripts j, k, l, m occurs exactly once in any of the four available positions (second, third, fourth and fifth) in $R_{ijkl,m}$.

2) In the cyclic order of positions ($2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 2$) each of the subscripts j, k, l, m is followed exactly twice by another subscript and exactly once by two others. In this way j is followed twice by k , k by j , l by m and m by l , so the preference of double following is mutual and splits the subscripts j, k, l, m in two pairs: $\{j, k\}$ and $\{l, m\}$.

Now we can prove that already established identities for $R_{ijkl,m}$ form a complete set at a point.

Theorem 5.18. *Assume that we are given a tensor $R_{ijkl,m}$ at the origin $0 \in \mathbb{R}^n$. Then necessary and sufficient conditions for this tensor to be the first covariant derivative of the curvature tensor of a Fedosov manifold at a point are:*

$$(a_1) \quad R_{ijkl,m} = -R_{ijlk,m} ;$$

$$(b_1) \quad R_{ijkl,m} = R_{jikl,m} ;$$

$$(c_1) \quad R_{i(jkl),m} := R_{ijkl,m} + R_{iklj,m} + R_{iljk,m} = 0 ;$$

$$(d_1) \quad R_{ij(kl,m)} := R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0 .$$

$$(e_1) \quad R_{imkj,l} + R_{ijml,k} + R_{iljk,m} + R_{iklm,j} = 0 .$$

Proof. The conditions (a₁)–(e₁) are necessary as was established above. Let us prove that they are sufficient. This is done similarly to the proof of Theorem 5.13.

Let us introduce a hypothetical normal tensor A_{ijklm} at 0 by the formula (5.20). We should first check that the tensor A_{ijkl} satisfies the conditions from the part 2) of Theorem 4.5 with $r = 2$ i.e. the symmetries

$$(5.28) \quad A_{ijklm} = A_{ikjlm}, \quad A_{ijklm} = A_{ijkml} ,$$

the identity (4.10) and the identity (5.27). This can be done by substitution of the expression (5.20) and similar one with permuted subscripts into the desired identities and subsequent use of the properties (a₁)–(e₁). For example, to prove the first equality in (5.28) interchange j and k in (5.20) and subtract the right hand sides. We get then

$$\begin{aligned} -6(A_{ijklm} - A_{ikjlm}) &= R_{ijkl,m} - R_{ikjl,m} + R_{iljm,k} - R_{ilkm,j} = \\ (R_{ijkl,m} + R_{iklj,m}) - R_{ilmj,k} - R_{ilkm,j} &= -R_{iljk,m} - R_{ilmj,k} - R_{ilkm,j} = 0 , \end{aligned}$$

where we used (a₁), (c₁) and (d₁). Similar straightforward calculations lead to the second equality in (5.28), as well as to (4.10). The proof of (5.27) can be obtained by reversing the calculations in the proof of Proposition 5.16 if we note first that the “differentiated” identity (2.9) follows from (a₁)–(c₁) as in the proof of Proposition 2.4.

Now we can supplement the obtained tensors A_{ijklm} by arbitrary set of tensors $A_{ijk\alpha_1\dots\alpha_r}$, $r \neq 2$, satisfying the conditions of the second part of Theorem 4.5 (for example we can take $A_{ijk\alpha_1\dots\alpha_r} = 0$, $r \neq 2$). By Theorem 4.5 there exists a structure of Fedosov manifold near 0 in \mathbb{R}^n with the normal tensors prescribed in this way.

It remains to check that the obtained Fedosov manifold indeed has the prescribed first covariant derivative of the curvature tensor at 0 i.e. that (5.4) holds with A_{ijklm} as constructed above and with $R_{ijkl,m}$ coinciding with the given initial tensor. This is also straightforward (the properties (a₁)–(d₁) should be used). \square

Remark 5.19. Clearly we can prescribe arbitrarily both tensors R_{ijkl} and $R_{ijkl,m}$ at 0 provided they satisfy the conditions of Theorems 5.13 and 5.18.

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