REMARKS ON TOPOLOGY OF THE HILBERT GRASSMANIAN.

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1. Introduction.

Topology of the usual grassmanians in finite-dimensional spaces plays an important role in analysis and topology. In particular it gives rise to characteristic classes of vector bundles and is also connected with important analytic objects like generalized hypergeometric functions.

In this paper we make some elementary remarks about some aspects of topology of the Hilbert grassmanian i.e. the grassmanian of the closed linear subspaces in a separable Hilbert space. It occurs that some interesting features can be found already on the level of general topology. We identify the points of this grassmanian with the self-adjoint projections in the given space. Then we can consider norm topology, strong and weak topology on the set of projections. The last two topologies in fact coincide as the topologies on the grassmanian and they define a separable metrizable topology there. However if we take closures of the set of self-adjoint projections in strong and weak topologies in the set of all bounded linear operators then the results will be different: the set of projections is closed in the strong topology but it is not closed in the weak topology. The closure in the weak topology is the "operator segment" $[0, 1]$: the set of all operators A such that $0 \leq A \leq I$ where I is the identity operator and the inequality is understood in the standard operator sense (as the inequality of the corresponding quadratic forms). This operator segment is a compact set in the weak operator topology.

We also consider a natural splitting of the grassmanian by the dimensions of the image and the kernel of the projection (unlike the finite-dimensional situation it is not sufficient to take the dimension of the image only: if the image is infinite-dimensional, then the kernel still can have any dimension). We describe the closures of the corresponding subsets in all topologies.

The weak topology was first introduced by J. von Neumann ([Ne]) who made an extensive use of it in the theory of operator algebras (see e.g. [Na]).

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A possible motivation to study topology of the Hilbert grassmanian is that it might give an approach to the invariant subspace problem. Namely, any bounded linear operator in the Hilbert space acts on the grassmanian and the problem reduces to finding a fixed point of this action in the set of non-trivial subspaces (subspaces which have non-vanishing dimension and codimension). I can only hope that the results of this paper can be ε-helpful in a future solving of the invariant subspace problem.

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1.Norm topology.

Denote by $\mathcal{H}$ a separable complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators in $\mathcal{H}$. We will denote $\|\cdot\|$ the standard operator norm in $\mathcal{B}(\mathcal{H})$.

Denote by $G_{n,k}(\mathcal{H})$ the set of all closed linear subspaces $L \subset \mathcal{H}$ such that $\dim L = n$ and $\dim L^{\perp} = k$ where n, k are non-negative integers or infinity and $n + k = \infty$. In particular $G_{0,\infty}(\mathcal{H}) = \{0\}$ and $G_{\infty,0}(\mathcal{H}) = \{\mathcal{H}\}$ consist each of one element only. Denote $G(\mathcal{H}) = \cup\{G_{n,k}(\mathcal{H}) : n + k = \infty\}$

Let us identify $L \in G_{n,k}(\mathcal{H})$ with the self-adjoint projection in $\mathcal{H}$ with the image L; denote this projection by P_L. Denote $\mathcal{P}_{n,k}$ the set of all self-adjoint projections corresponding to the subspaces $L \in G_{n,k}(\mathcal{H})$. So $\mathcal{P}_{n,k} \subset \mathcal{B}(\mathcal{H})$. Denote $\mathcal{P} = \cup\{\mathcal{P}_{n,k} : n + k = \infty\}$.

Lemma 1.1. *$\mathcal{P}_{n,k}$ is closed in $\mathcal{B}(\mathcal{H})$ in the norm topology.*

It follows that $G_{n,k}$, identified with $\mathcal{P}_{n,k}$, is a complete metric space with the metric $d(L, L') = \|P_L - P_{L'}\|$.

Proof of Lemma 1.1. Clearly $\mathcal{P}$ is closed in $\mathcal{B}(\mathcal{H})$ because its elements are distinguished in $\mathcal{B}(\mathcal{H})$ by the conditions $P^2 = P = P^*$. To check that $\mathcal{P}_{n,k}$ is closed, it is sufficient to prove that each $\mathcal{P}_{n.k}$ is open in $\mathcal{P}$. This follows if we use the following well known fact: if $P_1, P_2 \in \mathcal{P}$ and $\|P_1 - P_2\| < 1$ then $\dim P_1\mathcal{H} = \dim P_2\mathcal{H}$ (see e.g.[G-K]). $\square$

Denote $U(\mathcal{H})$ the group of all unitary operators in $\mathcal{H}$ (with norm topology). This is a complete metric space. Obviously $U(\mathcal{H})$ acts transitively on each of the grassmanians $G_{n,k}(\mathcal{H})$ with the stationary subgroup of a point $L \in G_{n,k}$ equal to $U(L) \times U(L^{\perp})$. Therefore for any fixed $L \in G_{n.k}(\mathcal{H})$ we have an isomorphism (of sets)

$$G_{n,k}(\mathcal{H}) = U(\mathcal{H})/[U(L) \times U(L^{\perp})] \tag{1.1}$$

where the quotient on the right means the set of the left cosets. We shall see that this is true with the topology as well.

Proposition 1.2. *Let us fix a subspace $L_0 \in G_{n,k}(\mathcal{H})$. Then the map*

$$\pi : U(\mathcal{H}) \longrightarrow G_{n,k}(\mathcal{H}), \quad U \mapsto UL_0,$$

is a locally trivial fiber bundle with the fiber $U(L_0) \times U(L_0^{\perp})$.

Proof. 1°. First note that the continuity of π follows from the fact that for any $L = UL_0,\ U \in U(\mathcal{H})$, we have $P_L = UP_{L_0}U^{-1} = UP_{L_0}U^*$.

2°. Suppose that for any $M \in G_{n,k}(\mathcal{H})$ we can construct a local section given in a neighbourhood $\mathcal{U}$ of M i.e. a continuous map $s : \mathcal{U} \to \pi^{-1}(\mathcal{U}) \subset U(\mathcal{H})$ such that $\pi \circ s = \mathrm{Id}_{\mathcal{U}}$. A trivialization of $\pi^{-1}(\mathcal{U})$ can be constructed then by

$$\mathcal{U} \times [U(L_0) \times U(L_0^{\perp})] \longrightarrow \pi^{-1}(\mathcal{U}), \quad (L, u) \mapsto s(L)u.$$

This map is obviously one-one and continuous. The inverse map

$$\pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times [U(L_0) \times U(L_0^{\perp})], \quad V \mapsto (\pi(V), [s(\pi(V))]^{-1}V)$$

is obviously continuous too. Therefore to prove the Proposition it is sufficient to construct local sections.

3°. Let us construct a local section near L_0. We will do it on the set

$$\mathcal{U} = \{L : \, d(L, L_0) < 1\} = \{L : \, \|P_L - P_{L_0}\| < 1\}.$$

Denoting $s(L) = U_L$ we see that U_L should be a unitary operator U_L such that it depends continuously on L and $U_L L_0 = L$. Let us take

$$A_L = P_L P_{L_0} + (I - P_L)(I - P_{L_0}).$$

Let us check that A_L defines linear topological isomorphisms $L_0 \to L$ and $L_0^{\perp} \to L^{\perp}$. Since A_L maps L_0 to L and $L_0^{\perp}$ to $L^{\perp}$, it is sufficient to prove that $P_L P_{L_0}$ defines a linear topological isomorphism $L_0 \to L$ and then replace P_L and P_{L_0} by $I - P_L$ and $I - P_{L_0}$ respectively. Now $P_L P_{L_0} = P_L$ on L_0 and $P_{L_0} P_L = P_{L_0}$ on L, so it is sufficient to prove that the operator $P_L P_{L_0} : L \to L$ is invertible. Obviously $P_L P_{L_0} = I + P_L(P_{L_0} - P_L)$ on L, so $\|P_L(P_{L_0} - P_L)\| < 1$ implies the desired invertibility.

Let us construct the polar decomposition of A_L. We have

$$A_L^* = P_{L_0} P_L + (I - P_{L_0})(I - P_L),$$

$$A_L^* A_L = P_{L_0} P_L P_{L_0} + (I - P_{L_0})(I - P_L)(I - P_{L_0}),$$

so $A_L^* A_L$ is an invertible selfadjoint operator which maps L_0 to L_0 and $L_0^{\perp}$ to $L_0^{\perp}$ isomorphically. Besides A_L and $A_L^* A_L$ depend continuously on L. Denote as usual $|A_L| = \sqrt{A_L^* A_L}$. Then $|A_L|$ is also invertible, depends continuously on L and leaves both spaces L_0 and $L_0^{\perp}$ invariant. Now we can take $U_L = A_L |A_L|^{-1}$. Then U_L is a unitary operator which maps L_0 to L and depends continuously on L, so it gives us the desired local section.

4°.To construct a section near an arbitrary point $M \in G_{n,k}(\mathcal{H})$ it is sufficient to use the homogeneity of the situation. □

Corollary 1.3. *$G_{n,k}$ is a connected component of $G(\mathcal{H})$.*

Corollary 1.4. *The topology of $G_{n.k}(\mathcal{H})$ is the quotient-topology of $U(\mathcal{H})$.*

Proof. We have to check that a set $\mathcal{U} \subset G_{n.k}(\mathcal{H})$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in $U(\mathcal{H})$. If $\mathcal{U}$ is open then $\pi^{-1}(\mathcal{U})$ is open because π is continuous. Vice versa suppose that $\pi^{-1}(\mathcal{U})$ is open. Then together with each point u_0 it contains its neighbourhood which has the form $\mathcal{U}_0 \times V_0$ in a trivialization of the Proposition 1.2. Here $\mathcal{U}_0$ is open in $G_{n,k}(\mathcal{H})$ and V_0 is open in $U(\pi(u_0)) \times U(\pi(u_0)^{\perp})$. But then $\mathcal{U} = \pi(\pi^{-1}(\mathcal{U}))$ contains the neighbourhood $\mathcal{U}_0$ of $\pi(u_0)$ as required. □

2. Strong and weak topology.

A. The strong and weak topologies on $G(\mathcal{H})$ and on its subsets are induced by the corresponding operator topologies on projections. We shall denote ${}^sG$ and ${}^wG$ the topological spaces obtained from $G(\mathcal{H})$ by supplying this set with strong and weak topology respectively. Similar meaning have notations ${}^sG_{n,k}$ and ${}^wG_{n,k}$.

Let us describe the strong and weak topologies more explicitely. We shall start with the strong topology.

Neighbourhoods in ${}^sG_{n,k}$: a base of neighbourhoods of $L_0 \in {}^sG_{n,k}$ can be chosen as follows:

$$ {}^s\mathcal{U}_{x_1,\ldots,x_N,\varepsilon}(L_0) = \{L : \|(P_L - P_{L_0})x_j\| < \varepsilon\}, \ j = 1, \ldots, N. $$

Here $x_j \in \mathcal{H}, \ \varepsilon > 0$.

It is clear that we can restrict ourselves to x_j with $\|x_j\| \leq 1$ (or even with $\|x_j\| = 1$). Moreover, we can restrict ourselves to the vectors from any dense subset $X \subset \mathcal{H}$ (or a subset X which is dense in a unit ball or unit sphere of $\mathcal{H}$). Indeed it is sufficient to prove that any neighbourhood of L_0 contains a neighbourhood of the form

$$ {}^s\mathcal{U}_{\tilde{x}_1,\ldots,\tilde{x}_N,\tilde{\varepsilon}}(L_0) \ \text{ with } \ \tilde{x}_j \in X. $$

But for the inclusion

$$ {}^s\mathcal{U}_{\tilde{x}_1,\ldots,\tilde{x}_N,\tilde{\varepsilon}}(L_0) \subset {}^s\mathcal{U}_{x_1,\ldots,x_N,\varepsilon}(L_0) $$

to be true it is suficient that for every $L \in {}^s\mathcal{U}_{\tilde{x}_1,\ldots,\tilde{x}_N,\tilde{\varepsilon}}(L_0)$

$$ \|(P_L - P_{L_0})x_j\| \leq \|(P_L - P_{L_0})\tilde{x}_j\| + \|(P_L - P_{L_0})(x_j - \tilde{x}_j)\| < \tilde{\varepsilon} + 2\|x_j - \tilde{x}_j\| \leq \varepsilon. $$

Hence we can choose $\tilde{\varepsilon} = \varepsilon/2$ and and take $\tilde{x}_j$ so that $\|x_j - \tilde{x}_j\| < \varepsilon/4$. It follows that

$$ {}^s\mathcal{U}_{\tilde{x}_1,\ldots,\tilde{x}_N,\varepsilon/2}(L_0) \subset {}^s\mathcal{U}_{x_1,\ldots,x_N,\varepsilon}(L_0) \ \text{ if } \ \|x_j - \tilde{x}_j\| < \varepsilon/4, \ j = 1, \ldots, N. $$

Furthermore, linearity implies that instead of a dense subset we can take X to be the set $\{e_j, \ j = 1, 2, \ldots\}$ which is a fixed orthonormal basis of $\mathcal{H}$. Hence we get the following base of neighbourhoods of L_0:

$$ {}^s\mathcal{U}_{N,\varepsilon}(L_0) = \{L : \|(P_L - P_{L_0})e_j\| < \varepsilon \ \ j = 1, \ldots, N\}. $$

Indeed, suppose that $x_j = \sum_{1 \leq k \leq M} c_{jk}e_k, \ c_{jk} \in \mathbf{C}$ and $\|x_j\| \leq 1$, $j = 1, \ldots, N$. Then $|c_{jk}| \leq 1$ and

$$ \|(P_L - P_{L_0})x_j\| \leq \sum_{1 \leq k \leq M} \|(P_L - P_{L_0})e_k\|. $$

It follows that

$$ {}^s\mathcal{U}_{M,\varepsilon/M}(L_0) \subset {}^s\mathcal{U}_{x_1,\ldots,x_N,\varepsilon}(L_0). $$

Summing up we get the following

Lemma 2.1. *In ${}^sG_{n,k}$ every point L_0 has a countable base of neighbourhoods of the form ${}^s\mathcal{U}_{N,1/N}(L_0)$, $N = 1, 2, \ldots$. The strong convergence $s\text{-}\lim_\alpha L_\alpha = L$ is equivalent to the convergence $\lim_\alpha P_{L_\alpha} e_j = P_L e_j$, $j = 1, 2, \ldots$ where $\{e_j,\ j = 1, 2, \ldots\}$ is a fixed orthonormal basis in $\mathcal{H}$.*

Corollary 2.2. *${}^sG_{n,k}$ is metrizable.*

Proof. Identifying L with the sequence

$$\{P_L e_j,\ j = 1, 2, \ldots\} \in \mathcal{H}^\infty = \mathcal{H} \times \mathcal{H} \times \ldots,$$

we get an embedding $G_{n,k}(\mathcal{H}) \subset \mathcal{H}^\infty$. There is a metric on $\mathcal{H}^\infty$ given by

$$d(\{f_j\}, \{h_j\}) = \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{\|f_j - h_j\|}{1 + \|f_j - h_j\|}.$$

It is easy to see that this metric induces a metric on $G_{n,k}(\mathcal{H})$ such that the corresponding topology coincides with the strong topology. $\square$

Remark. Unlike the topology on ${}^sG_{n,k}$ the metric is not unitary invariant.

Corollary 2.3. *${}^sG_{n,k}$ is separable, i.e. has a countable dense subset.*

Proof. It is suficient to notice that $\mathcal{H}^\infty$ is separable. $\square$

B. Now let us describe the weak topology in $G_{n,k}(\mathcal{H})$.

Neighbourhoods in ${}^wG_{n,k}$: a base of neighbourhoods of $L_0 \in^w G_{n,k}$ can be taken as

$${}^w\mathcal{U}_{x_1,\ldots,x_N,\varepsilon}(L_0) = \{L :\ |((P_L - P_{L_0})x_j, x_k)| < \varepsilon\}, \quad j, k = 1, \ldots, N,$$

where $x_j \in \mathcal{H}$, $\varepsilon > 0$. Again we can restrict ourselves to the vectors x_j from a dense subset X in $\mathcal{H}$ or in the unit ball of $\mathcal{H}$. Moreover if is sufficient to take $x_j = e_j$ where $\{e_j,\ j = 1, 2, \ldots\}$ is an orthonormal basis in $\mathcal{H}$. Therefore we get a base of neighbourhoods of each point L_0 of the form

$${}^w\mathcal{U}_{N,\varepsilon}(L_0) = \{L :\ |((P_L - P_{L_0})e_j, e_k)| < \varepsilon\}, \quad j, k = 1, \ldots, N.$$

Therefore we get the following

Lemma 2.4. *In ${}^wG_{n,k}$ every point L_0 has a countable base of neighbourhoods of the form ${}^w\mathcal{U}_{N,1/N}(L_0)$, $N = 1, 2, \ldots$. The weak convergence $w\text{-}\lim_\alpha L_\alpha = L$ is equivalent to the convergence of the matrix elements of the operators P_{L_α} to the matrix elements of P_L in the basis e_j, $j = 1, 2, \ldots$.*

Corollary 2.5. *${}^wG_{n,k}$ is metrizable and separable.*

Now note that exactly in the same way we can describe the strong and weak topologies on $G(\mathcal{H})$. The arguments do not change and we come to

Lemma 2.6. *In ${}^sG$ and in ${}^wG$ every point L_0 has a countable base of neighbourhoods of the form ${}^s\mathcal{U}_{N,1/N}(L_0)$ and ${}^w\mathcal{U}_{N,1/N}(L_0)$ respectively, $N = 1, 2, \ldots$. The convergence $s\text{-}\lim_\alpha L_\alpha = L$ and $w\text{-}\lim_\alpha L_\alpha = L$ in ${}^sG$ and ${}^wG$ is described exactly the same way as in ${}^sG_{n,k}$ and ${}^wG_{n,k}$.*

Corollary 2.7. *${}^sG$ and ${}^wG$ are metrizable and separable.*

3. Comparison of strong and weak topologies.

A. In this section we shall compare the strong and weak topologies on $G_{n,k}(\mathcal{H})$ and $G(\mathcal{H})$ and describe the closures of these sets in $\mathcal{B}(\mathcal{H})$ in both topologies.

Proposition 3.1. *Strong and weak topologies coincide on $G_{n,k}(\mathcal{H})$ and $G(\mathcal{H})$.*

Proof. Due to Corollaries 2.2, 2.3, 2.5 and 2.7 it is sufficient to check that for any sequence $P_n \in \mathcal{P},\ n = 1, 2, \dots$

$$w\text{-}\lim_{n\to\infty} P_n = P \in \mathcal{P} \ \text{ implies } \ s\text{-}\lim_{n\to\infty} P_n = P.$$

This fact is well known (see [H1], Problem 115) and follows from the fact that $f_n \to f$ (weak) and $\|f_n\| \to \|f\|$ imply $f_n \to f$ (strong) ([H1], Problem 20). □

Remark. This proposition is basically well known as well as the arguments which are used in its proof. Similar results on measure-preserving transformations and corresponding arguments for general unitary operators can be found in [H2] and [Na] respectively.

In the proof of Proposition 3.1 we assumed that the weak limit of the projections P_n is again a projection. This is not always true - see e.g. [H1], Problem 115. But this is true for strong convergence, even if we replace sequences by nets:

Lemma 3.2. *Let $\{P_\alpha\}$ be a net of projections $P_\alpha \in \mathcal{P}$ and there exists a strong limit $s\text{-}\lim_\alpha P_\alpha = P$. Then $P \in \mathcal{P}$ i.e. P is a self-adjoint projection too.*

Proof. Taking limit in the equality $(P_\alpha u, v) = (u, P_\alpha v),\ u, v \in \mathcal{H}$, we get $(Pu, v) = (u, Pv)$, i.e. P is self-adjoint. We also have

$$(Pu, u) = \lim_\alpha (P_\alpha u, u) \in [0, 1].$$

Therefore $0 \le P \le I$ in the operator sence. It follows that $\|P\| \le 1$. Now writing

$$P_\alpha^2 u - P^2 u = P_\alpha(P_\alpha - P)u + (P_\alpha - P)Pu, \quad u \in \mathcal{H},$$

we immediately see that $\|P_\alpha^2 u - P^2 u\| \to 0$. Hence $P_\alpha^2 = P_\alpha$ implies $P^2 = P$. □

Proposition 3.3. *Weak closure of the set $\mathcal{P}$ of all self-adjoint projections in $\mathcal{B}(\mathcal{H})$ coincides with the set $[\mathbf{0}, \mathbf{I}]$ of all self-adjoint operators $A \in \mathcal{B}(\mathcal{H})$ such that $0 \le A \le I$.*

Proof. Let us denote the strong and weak closures of any subset $M \subset \mathcal{B}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ by $\mathrm{Cl}_s(M)$ and $\mathrm{Cl}_w(M)$ respectively. Suppose that $A \in \mathrm{Cl}_w(\mathcal{P})$ i.e. $A \in \mathcal{B}(\mathcal{H})$ is a weak limit of self-adjoint projections. The first part of the proof of Lemma 3.2 shows that $A \in [\mathbf{0}, \mathbf{I}]$. This means that $\mathrm{Cl}_w(\mathcal{P}) \subset [\mathbf{0}, \mathbf{I}]$.

Let us prove the inverse inclusion. We have to show that every $A \in [\mathbf{0}, \mathbf{I}]$ is a weak limit of self-adjoint projections. Let us choose an orthonormal basis $e_1, e_2, \dots$ in $\mathcal{H}$ and denote by P_n the self-adjoint projection of rank n on the subspace spanned by $e_1, e_2, \dots, e_n$. Then $P_n A P_n \to A$ strongly, and $P_n A P_n \in [\mathbf{0}, \mathbf{I}]$ for all n. Therefore it is sufficient to check the inclusion $A \in \mathrm{Cl}_w(\mathcal{P})$ for operators of finite rank from $[\mathbf{0}, \mathbf{I}]$. But by the spectral theorem every such operator is a finite sum of the operators of the form $\alpha^2 P_1$ where P_1 is a self-adjoint projection of rank 1 and $\alpha \in [0,1]$. Therefore it remains to prove that $\alpha^2 P_1 \in \mathrm{Cl}_w(\mathcal{P})$. To do this we shall practically follow the solution of the Problem 115 in [H1].

Let us choose an orthonormal basis $e_1, e_2, \dots$ of $\mathcal{H}$ so that the image of P_1 is spanned by e_1. Now let us consider self-adjoint projections $P_{(n)}$ given by

$$P_{(n)}u = (u, \alpha e_1 + \beta e_n)(\alpha e_1 + \beta e_n), \quad u \in \mathcal{H},$$

where $\beta \in [0,1]$ is chosen so that $\alpha^2 + \beta^2 = 1$. We have then for any $u, v \in \mathcal{H}$

$$\lim_{n\to\infty} (P_{(n)}u, v) = \lim_{n\to\infty} (u, \alpha e_1 + \beta e_n)(\alpha e_1 + \beta e_n, v) = \alpha^2 (u, e_1)(e_1, v) = \alpha^2 (P_1 u, v),$$

which exactly means that $P_{(n)} \to \alpha^2 P_1$ weakly as $n \to \infty$. $\square$

Remark. The set $[\mathbf{0}, \mathbf{I}]$ considered in the weak operator topology, is a compact topological space.

Indeed we can identify elements $A \in [\mathbf{0}, \mathbf{I}]$ with their matrices $[a_{ij}]$ in a fixed orthonormal basis $e_1, e_2, \dots$. (Here $a_{ij} = (Ae_i, e_j)$.) Then obviously $a_{ii} \in [0,1]$ and also $|a_{ij}| \le 1$ due to the Cauchy-Schwarz inequality $|a_{ij}|^2 \le a_{ii}a_{jj}$. Let K be the unit disc in the complex plane $\mathbf{C}$. Then we obtain a natural inclusion of $[\mathbf{0}, \mathbf{I}]$ in a product of a countable number of copies K_{ij} of K, identifying every element $A \in [\mathbf{0}, \mathbf{I}]$ with the set of its matrix elements. If we introduce the Tikhonov topology on the product then it becomes a compact topological space and $[\mathbf{0}, \mathbf{I}]$ becomes its closed subset, hence also a compact topological space.

B. Now we shall describe strong and weak closures of the sets $G_{n,k}(\mathcal{H})$ in $G(\mathcal{H})$ and also the weak closures of the corresponding sets of projections in $\mathcal{B}(\mathcal{H})$.

Proposition 3.4. *The strong and weak closure of $G_{n,k}(\mathcal{H})$ in $G(\mathcal{H})$ coincides with*

$$\cup\{G_{n',k'} : \ n' \le n, \ k' \le k, \ n' + k' = \infty\}.$$

In other words

(i) $\mathrm{Cl}_s(G_{\infty,\infty}(\mathcal{H})) = \mathrm{Cl}_w(G_{\infty,\infty}(\mathcal{H})) = G(\mathcal{H})$;

(ii) $\mathrm{Cl}_s(G_{n,\infty}(\mathcal{H})) = \mathrm{Cl}_w(G_{n,\infty}(\mathcal{H})) = \cup_{p=0}^{n} G_{p,\infty}$ *if n is finite;*

(iii) $\mathrm{Cl}_s(G_{\infty,n}(\mathcal{H})) = \mathrm{Cl}_w(G_{\infty,n}(\mathcal{H})) = \cup_{q=0}^{n} G_{\infty,q}(\mathcal{H})$ *if n is finite.*

Proof. We have seen already that $\mathrm{Cl}_s(G_{n,k}(\mathcal{H})) = \mathrm{Cl}_w(G_{n,k}(\mathcal{H}))$ (if the closures are taken in $\mathcal{P}$).

Let us check that for every $P \in \mathrm{Cl}_w(\mathcal{P}_{n,k}) \cap \mathcal{P}$ we have

$$\dim P\mathcal{H} \le n, \quad \text{and} \quad \dim(I - P)\mathcal{H} \le k.$$

We can find a sequence $\{P_j | \, j = 1, 2, \dots\}$ such that $P_j \in \mathcal{P}_{n,k}$ for all j and $P_j \to P$ weakly as $j \to \infty$. Let us consider the diagonal elements of P_j. Applying the Fatou Lemma we get

$$\dim P\mathcal{H} = \operatorname{Tr} P \le \liminf_{j\to\infty} \operatorname{Tr} P_j = \operatorname{Tr} P_j = n.$$

Similarly we can check that $\dim(I - P) \le k$.

Conversely suppose that $\dim P\mathcal{H} \le n$ and $\dim(I - P)\mathcal{H} \le k$. Let us prove that $P \in \mathrm{Cl}_s(\mathcal{P}_{n,k})$. Suppose that $\dim P\mathcal{H} = n' < \infty$ (all other cases are considered similarly). Then $k = \infty$. If $n' = n$, then $P \in \mathcal{P}_{n,k}$ and we have nothing to prove. Therefore assume that $n' < n$. Let us choose an orthonormal basis $\{e_1, e_2, \dots\}$ in $\mathcal{H}$ so that $e_1, \dots, e_{n'}$ span $P\mathcal{H}$. Now let us take a subspace $L_N \subset \mathcal{H}$ with $\dim L_N = n$ spanned by $e_1, e_2, \dots, e_{n'}, e_{N+1}, e_{N+2}, \dots, e_{N+n-n'}$. Denote by P_N the self-adjoint projection in $\mathcal{H}$ with the image L_N. Then $P_N \in \mathcal{P}_{n,k}$ and $P_N \to P$ strongly as $N \to \infty$. $\square$

Now we shall consider the weak closure of $\mathcal{P}_{n,k}$ in $\mathcal{B}(\mathcal{H})$.

Proposition 3.5. *The weak closure* $\mathrm{Cl}_w(\mathcal{P}_{n,k})$ *of* $\mathcal{P}_{n,k}$ *in* $\mathcal{B}(\mathcal{H})$ *coincides with the set*

$$\mathcal{A}_{n,k} = \{A : \; A \in [\mathbf{0}, \mathbf{I}], \; \operatorname{rank} A \le n, \; \operatorname{rank}(I - A) \le k\},$$

where $\operatorname{rank} A = \dim \operatorname{Im} A$.

Proof. (i) Let us first consider the case $n = k = \infty$. Then the result follows from Proposition 3.3 and Proposition 3.4 (i).

(ii) Now let us assume that $n < \infty$ (the case $k < \infty$ is treated similarly), so

$$\mathcal{A}_{n,k} = \mathcal{A}_{n,\infty} = \{A : \; A \in [\mathbf{0}, \mathbf{I}], \; \operatorname{rank} A \le n\}.$$

Let us prove that $\mathrm{Cl}_w(\mathcal{P}_{n,k}) \subset \mathcal{A}_{n,k}$. Suppose that $P_j \in \mathcal{P}_{n,k}, \; j = 1, 2, \dots$, $\operatorname{rank} P_j \le n$ for all j and there exists a weak limit A of P_j as $j \to \infty$. The inclusion $A \in [\mathbf{0}, \mathbf{I}]$ follows from Proposition 3.3, so we only have to check that $\operatorname{rank} A \le n$.

Note first that $\operatorname{Tr} A \le n$ by the argument from the proof of Proposition 3.4. Therefore A is in the trace class, hence compact. We can then find an orthonormal basis of eigenvectors of A. Denote the vectors of this basis $e_1, e_2, \dots$. The eigenvalues of A are non-negative and have 0 as their only accumulation point. Therefore we can put them into the non-increasing order. The rank of A is equal to the number of non-vanishing eigenvalues, multiplicities counted.

Assume, arguing by contradiction, that $\operatorname{rank} A \ge n + 1$. Then in the chosen basis A will be represented by a diagonal matrix with positive first $n + 1$ diagonal elements. For large j the left-upper-corner $(n + 1) \times (n + 1)$ minor in the matrix of P_j will be non-vanishing. But this obviously implies that $\operatorname{rank} P_j \ge n + 1$ because then $P_j e_1, \dots, P_j e_{n+1}$ will be linearly independent. This contradicts the assumption $P_j \in \mathcal{P}_{n,k}$ and proves the inequality $\operatorname{rank} A \le n$.

(iii) It remains to prove that for any $A \in \mathcal{A}_{n,\infty}$ with $n < \infty$ we can find $P_N \in \mathcal{P}$, $N = 1, 2, \dots$ with $\operatorname{rank} P_N \le n$ so that $P_N \to A$ weakly as $N \to \infty$. By the spectral theorem there exists an orthonormal basis $e_1, e_2, \ldots$ in $\mathcal{H}$ such that $Ae_j = \alpha_j^2 e_j$ with $0 \leq \alpha_j \leq 1$ for all j. This means that A can be written in the form

$$A = \sum_{j=1}^{n} \alpha_j^2 P^{(j)},$$

where $P^{(j)}$ is a self-adjoint projection of rank 1 acting by the formula $P^{(j)}u = (u, e_j)e_j$, $u \in \mathcal{H}$. We have seen in the proof of Proposition 3.3 that $\alpha_j^2 P^{(j)}$ is a weak limit of rank 1 self-adjoint projections i.e. projections from $\mathcal{P}_{1,\infty}$. Making a particular explicit choice of these projections we can write

$$P_{j,N} \to \alpha_j^2 P^{(j)} \text{ weakly as } N \to \infty,$$

where $P_{j,N}$ is given by the formula

$$P_{j,N}u = (u, \alpha_j e_j + \beta_j e_{N+j})(\alpha_j e_j + \beta_j e_{N+j}),$$

with $\beta_j \in [0, 1]$, $\alpha_j^2 + \beta_j^2 = 1$. Now for $N > n$ define

$$P_N = \sum_{j=1}^{n} P_{j,N}.$$

It is easy to check that $P_N \in \mathcal{P}_{n,\infty}$ and $P_N \to A$ weakly as $N \to \infty$. $\square$

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