

A sequence of connections and a characterization of Kähler manifolds

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Dedicated to Mel Rothenberg on the occasion of his 65th birthday.

ABSTRACT. We study a sequence of connections which is associated with a Riemannian metric and an almost symplectic structure on a manifold M according to a construction in [5]. We prove that if this sequence is trivial (i.e. constant) or 2-periodic, then M has a canonical Kähler structure.

Introduction

In this paper we consider a sequence of connections which is associated with two structures on the same manifold M : a Riemannian metric g and an almost symplectic structure ω . This sequence was introduced in [5]. We discuss a special situation when this sequence is in fact trivial, i.e. contains only one connection. We prove that in this case M is a Kähler manifold and the Kähler structure is obtained from the structures (g, ω) by a simple procedure which includes a Gromov construction of an almost complex structure J which is compatible with ω . In our case J proves to be integrable and together with ω (which proves to be closed) they generate a Kähler structure.

We also note that if the sequence of connections has period 2, then it is in fact trivial, so we again have a Kähler structure.

In Sections 1 and 2 we recall the construction of the sequence of connections from [5] and the Gromov construction of an almost complex structure associated with a pair (g, ω) . The characterization of the Kähler manifolds is given in Sect.3.

It is well known that the existence of a Kähler structure imposes many topological restrictions on M . In particular, there are symplectic manifolds (M, ω) which do not admit any Kähler structure, even with a different ω (see e.g. [2], p.147, for a Thurston example, and also the book [9] for many other examples).

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It would be interesting to provide topological obstructions to the existence of a pair of forms (g, ω) such that the corresponding sequence of connections is periodic with a specific period. The simplest new situations arise when this sequence is trivial but the initial connection is not symmetric or when the sequence (starting with the Levi-Civita connection of g) has period 2 not from the very beginning but from some further term.

All periodic situations give rise to new classes of manifolds which are generalizations of Kähler manifolds. It is quite probable that they are of geometric interest.

1. Sequence of connections associated with two forms.

Let M be a C^∞ manifold, $\dim M = n$, g is a pseudo-riemannian structure on M , i.e. a non-degenerate symmetric form on each tangent space $T_x M$, $x \in M$, which smoothly depends on x . In local coordinates (x^1, \dots, x^n) such a form is given by a non-degenerate symmetric tensor $g_{ij} = g(\partial_i, \partial_j)$, where $\partial_j = \partial/\partial x_j$ is a basic vector field in the domain U of the local coordinates. Here $g_{ij} = g_{ij}(x)$ is a C^∞ function on U .

Let ∇ be a linear connection i.e. a connection in the tangent bundle TM . It is given locally by the Christoffel symbols Γ_{ij}^k defined from the relation $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$, where we use the standard summation convention. We will identify the connection ∇ with the set of its Christoffel symbols, and will talk about a connection Γ (instead of ∇) which does not lead to a confusion.

The torsion tensor of the connection ∇ is given by $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$.

The connection ∇ preserves g if

$$(1.1) \quad Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

where X, Y, Z are arbitrary vector fields on M .

It is well known due to T. Levi-Civita, H. Weyl and E. Cartan (see [3]) that for any tensor $T = (T_{ij}^k)$, which is antisymmetric with respect to the subscripts i, j , there exists a unique connection $\Gamma = (\Gamma_{ij}^k)$ with the torsion tensor T (i.e. $\Gamma_{ij}^k - \Gamma_{ji}^k = T_{ij}^k$) such that Γ preserves g . If $T = 0$, this connection is called the *Levi-Civita connection*. It is the *canonical* connection associated with g .

A skew-symmetric (or rather almost symplectic) analog of this fact was established in [5]. Namely, let ω be an almost symplectic structure on M i.e. a non-degenerate 2-form on M or skew-symmetric bilinear form on each tangent space $T_x M$ which is smooth in x . In local coordinates such a form is determined by a non-degenerate skew-symmetric tensor $\omega_{ij} = \omega(\partial_i, \partial_j)$. A connection ∇ (or Γ) on TM preserves ω if an identity similar to (1.1) holds, namely:

$$(1.2) \quad X\omega(Y, Z) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z),$$

for any vector fields X, Y, Z on M .

Now for any connection $\Gamma = (\Gamma_{ij}^k)$ on M denote by $\Pi = (\Pi_{ij}^k)$ its symmetric part ($\Pi = \Pi(\Gamma)$) i.e.

$$(1.3) \quad \Pi_{ij}^k = \frac{1}{2}(\Gamma_{ij}^k + \Gamma_{ji}^k).$$

THEOREM 1.1. [5] *The map $\Gamma \mapsto \Pi(\Gamma)$ gives a bijective affine correspondence between the set of all connections preserving ω and the set of all symmetric connections. The inverse map $\Pi \mapsto \Gamma$ is given by*

$$(1.4) \quad \Gamma_{kij} = \frac{1}{2}(\partial_k \omega_{ij} - \partial_i \omega_{jk} - \partial_j \omega_{ki}) + (\Pi_{kij} + \Pi_{jik} - \Pi_{ijk}).$$

In case when ω is closed (hence symplectic) this formula can be rewritten as

$$(1.5) \quad \Gamma_{kij} = \partial_k \omega_{ij} + (\Pi_{kij} + \Pi_{jik} - \Pi_{ijk}).$$

REMARK 1.2. It is a well known fact that a symmetric connection preserving ω exists if and only if ω is closed (see e.g. [7], [8], [10], [5]).

Now we are ready to recall the construction of the sequence of connections from [5].

Suppose we are given non-degenerate symmetric and antisymmetric 2-forms g and ω at the same time. Our first (or rather 0th) connection ∇_0 (or Γ_0) will be just the Levi-Civita connection of g . It is symmetric (or its torsion is $T_0 = 0$), so we will also put $\Pi_0 = \Gamma_0$. The next connection ∇_1 (or Γ_1) will be the unique connection preserving ω and having the symmetric part $\Pi(\Gamma_1) = \Pi_0$. Denote its torsion T_1 . The next connection will be the unique connection ∇_2 (or Γ_2) which preserves g and has the torsion T_1 . Denote by Π_1 its symmetric part. Then we can take the connection ∇_3 (or Γ_3) which preserves ω and has the symmetric part Π_1 . In this way we obtain a sequence of connections

$$(1.6) \quad \Gamma_0, \Gamma_1, \Gamma_2, \dots,$$

which is uniquely defined by the following properties:

- (i) The connection Γ_{2k} preserves g for any $k = 0, 1, \dots$
- (ii) The connection Γ_{2k+1} preserves ω for any $k = 0, 1, \dots$
- (iii) The connections Γ_{2k} and Γ_{2k+1} have the same symmetric part for any $k = 0, 1, \dots$
- (iv) The connections Γ_{2k+1} and Γ_{2k+2} have the same torsion tensor for any $k = 0, 1, \dots$
- (v) The connection Γ_0 is symmetric.

Instead of requiring (v) we could also start with an arbitrary torsion-type tensor T_0 and take Γ_0 preserving g and having the torsion tensor T_0 . This gives us a sequence which also depends on the choice of T_0 .

Note that for Kähler manifolds, starting with $T_0 = 0$ we obtain the constant sequence i.e. we get $\Gamma_0 = \Gamma_1 = \Gamma_2 = \dots$. Our main goal here will be to prove the inverse statement under the condition that the metric g is in fact positive (i.e. Riemannian).

2. Gromov's almost complex structure construction

Let (M, ω) be an almost symplectic manifold. Let us recall

DEFINITION 2.1. An almost complex structure J in TM is called *compatible with the almost symplectic structure* ω , if

(i) $\omega(X, JX) > 0$ for any tangent vector $X \neq 0$.

(ii) $\omega(JX, Y) + \omega(X, JY) = 0$ for any tangent vectors $X, Y \in T_x M$, $x \in M$,
or, equivalently:

$$(2.1) \quad g(X, Y) = \omega(X, JY)$$

is an hermitian metric i.e. a Riemannian metric such that J is an isometry in this metric.

Note also that a condition equivalent to (ii) is

$$\omega(JX, JY) = \omega(X, Y).$$

PROPOSITION 2.1. (M. Gromov [4]) *On any almost symplectic manifold (M, ω) there exists an almost complex structure J which is compatible with ω .*

PROOF. In fact M. Gromov gave a construction of such an almost complex structure J in TM . This construction depends on a choice of an arbitrary start-up Riemannian metric g_0 . After this choice the construction of J becomes canonical. So in fact we need a linear algebra construction in each fiber of TM which is canonical in ω , g_0 (in particular smoothly depending on ω and g_0). Let us describe this construction following [1] (section 6.1.1).

Denote $V = T_x M$ where $x \in M$ is fixed, so V is a symplectic vector space. Define a linear endomorphism $A : V \rightarrow V$ by

$$g_0(AX, Y) = \omega(X, Y).$$

Then A is antisymmetric with respect to g_0 . Therefore $-A^2$ is symmetric and

$$g_0(-A^2 X, X) = -\omega(A X, X) = \omega(X, A X) = g_0(A X, A X) > 0,$$

provided $X \in V \setminus 0$. Therefore $-A^2$ is a symmetric positive definite operator. Define $B = \sqrt{-A^2}$ to be the symmetric positive-definite square-root of $-A^2$. This can be done canonically e.g. by using the Cauchy integral and choosing a branch of the square root in $\mathbb{C} \setminus (-\infty, 0]$ with $\sqrt{1} = 1$. Clearly B is invertible and commutes with A . Now take $J = B^{-1}A$. Then $J^2 = -\text{Id}_V$. It is also clear that J is antisymmetric with respect to g_0 . Furthermore,

$$\omega(X, JX) = g_0(A X, B^{-1}A X) > 0, \quad X \in V \setminus 0,$$

because B^{-1} is also symmetric and positive definite. We also have

$$\begin{aligned} \omega(JX, Y) &= g_0(A J X, Y) = g_0(A B^{-1}A X, Y) = -g_0(B^{-1}A X, A Y) \\ &= -g_0(A X, B^{-1}A Y) = -\omega(X, JY), \end{aligned}$$

which proves that J is compatible with ω .

It follows also that the formula (2.1) defines an hermitian metric g on M . \square

3. Main results

Let us assume that we have a triple (M, g_0, ω) , where M is a C^∞ -manifold, g_0 is a Riemannian metric on M , ω is an almost-symplectic form on M . Let ∇ (or Γ) denote the Levi-Civita connection associated with g_0 .

THEOREM 3.1. *Let us assume that the sequence of connections on M associated with M, g_0, ω as described in Sect.1, is trivial, i.e. ∇ preserves ω (as well as g_0). Then there exists an integrable complex structure J on M (i.e. on TM), such that if g is defined by (2.1), then g is an hermitian metric on M with the same Levi-Civita connection ∇ , and the triple (g, ω, J) defines a Kähler structure on M .*

PROOF. Note first that if $\nabla\omega = 0$ for a symmetric connection ∇ , then $d\omega = 0$ (see e.g. [7], [8], [10] and Remark 1.4 in [5]), so the form ω is closed, hence it is a symplectic form.

Now take the almost complex structure J and the Riemannian metric g which are constructed by the data (M, ω, g_0) as in the proof of Proposition 2.1. By definition the connection ∇ preserves both ω and g_0 (i.e. $\nabla\omega = \nabla g_0 = 0$). We claim that the same is true for J and g .

Indeed, J and g are constructed canonically from ω and g_0 . But the ∇ -parallel transport (which preserves both ω and g_0) preserves all the relations which are described in the proof of Proposition 2.1. Hence it preserves J and g . In particular, ∇ is the Levi-Civita connection for g (as well as for g_0).

Now we need to establish that J is integrable (i.e. J is a complex structure on M) and (g, ω, J) define a Kähler structure on M . This follows from the arguments given in [6], vol.2, Ch.IX, Sect.4 or [2], p.148. \square

THEOREM 3.2. *Assume that the sequence of connections associated with (g_0, ω) has period 2, i.e. $\Gamma_2 = \Gamma_0$. Then in fact we also have $\Gamma_0 = \Gamma_1 = \Gamma_2 = \dots$, i.e. the sequence of connections is trivial and M has a Kähler structure which is canonically defined by (g_0, ω) .*

PROOF. The relation $\Gamma_2 = \Gamma_0$ implies in particular that Γ_1 is symmetric, because Γ_1 and Γ_2 share the same torsion tensor. On the other hand Γ_0 and Γ_1 share the same symmetric part, so it follows that $\Gamma_0 = \Gamma_1$ because both Γ_0 and Γ_1 are symmetric. \square

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