THE MIURA MAP ON THE LINE

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Abstract. We study relations between properties of the Miura map $r \mapsto q = B(r) = r' + r^2$ and Schrödinger operators $L_q = -d^2/dx^2 + q$ where r and q are real-valued functions or distributions (possibly not decaying at infinity) from various classes. In particular, we study B as a map from $L^2_{\text{loc}}(\mathbb{R})$ to the local Sobolev space $H^{-1}_{\text{loc}}(\mathbb{R})$ and the restriction of B to the Sobolev spaces $H^\beta(\mathbb{R})$ with $\beta \geq 0$. For example, we prove that the image of B on $L^2_{\text{loc}}(\mathbb{R})$ consists exactly of those $q \in H^{-1}_{\text{loc}}(\mathbb{R})$ such that the operator L_q is positive. We also investigate mapping properties of the Miura map in these spaces. As an application we prove an existence result for solutions of the Korteweg-de Vries equation in $H^{-1}(\mathbb{R})$ for initial data in the range $B(L^2(\mathbb{R}))$ of the Miura map.

Contents

1. Introduction and Main Results

The Miura map is the nonlinear mapping

$$B(r) = r' + r^2 \tag{1.1}$$

which takes classical solutions of the modified Korteweg-de Vries (mKdV) equation to classical solutions of the Korteweg-de Vries (KdV) equation. More precisely, let

$$\text{mKdV}(v) := v_t - 6v^2 v_x + v_{xxx}$$

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and

$$\mathrm{KdV}(u) := u_t - 6uu_x + u_{xxx}$$

for functions $v, u \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R})$. Then

$$\mathrm{KdV}(B(v)) = (\mathrm{mKdV}(v))_x + 2v \cdot \mathrm{mKdV}(v) \tag{1.2}$$

so that $\mathrm{KdV}(B(v)) = 0$ whenever $mKdV(v) = 0$. The Miura map has been used extensively to relate existence and uniqueness results for solutions for the mKdV and KdV equations. More fundamentally, the global geometry of nonlinear differentiable mappings such as the Miura map has been studied to solve various nonlinear differential equations.

It is well-known that the KdV equation can be successfully studied with the help of the spectral theory of the Schrödinger operators

$$L_q := -d^2/dx^2 + q. \tag{1.3}$$

In particular, in appropriate classes of potentials q, the spectrum of L_q is preserved by the KdV flow applied to q. (See e.g. [32].) The Miura map is formally related with the Schrödinger operator as follows: the relation $q = B(r) = r' + r^2$ is equivalent to the following factorization of L_q:

$$L_q = L_{B(r)} = (\partial_x - r)^+(\partial_x - r) = (-\partial_x - r)(\partial_x - r), \tag{1.4}$$

where $\partial_x = d/dx$, and A^+ means the operator formally adjoint to an operator A in functions on $\mathbb{R}$ (with respect to the scalar product in $L^2(\mathbb{R})$).

The aim of this paper is to study the Miura map and its geometry on the real line with an emphasis on function spaces of low regularity. Our first result characterizes the range of the Miura map. We denote by B the map (1.1) from real-valued functions in $L^2_{\mathrm{loc}}(\mathbb{R})$ into the local Sobolev space $H^{-1}_{\mathrm{loc}}(\mathbb{R})$. If q is a real-valued distribution in $H^{-1}_{\mathrm{loc}}(\mathbb{R})$, the operator L_q maps $\mathcal{C}_0^\infty(\mathbb{R})$ into the Sobolev space $H^{-1}(\mathbb{R})$, so that $(L_q\varphi, \varphi)$ is well-defined for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. We write $L_q \geq 0$ if $(L_q\varphi, \varphi) \geq 0$ for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$.

Theorem 1.1. *Let q be a real-valued distribution belonging to $H^{-1}_{\mathrm{loc}}(\mathbb{R})$. Then the following statements are equivalent.*
(i) $q \in \mathrm{Im}(B)$, i.e., $q = r' + r^2$ for some real-valued function $r \in L^2_{\mathrm{loc}}(\mathbb{R})$.
(ii) The equation $L_q y = 0$ has a strictly positive solution $y \in H^1_{\mathrm{loc}}(\mathbb{R})$.
(iii) $L_q \geq 0$.

Next, we consider the restriction of B to the Sobolev space $H^\beta(\mathbb{R})$ for $\beta \geq 0$; we denote this restriction by B_β. Although B_β has range contained in $H^{\beta-1}(\mathbb{R})$, it is not true that $\mathrm{Im}(B_\beta) = \mathrm{Im}(B) \cap H^{\beta-1}(\mathbb{R})$; rather, an additional condition is needed to characterize its range.

Theorem 1.2. *Let $\beta \geq 0$ be arbitrary. A real-valued distribution $q \in H^{\beta-1}(\mathbb{R})$ belongs to $\mathrm{Im}(B_\beta)$ if and only if*
(i) $L_q \geq 0$, and
(ii) q can be presented in the form $q = f' + g$ for $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$.

In addition, we will give an alternative characterization of $\mathrm{Im}(B_\beta)$ in terms of a "special integral" of q on $\mathbb{R}$ which coincides with the ordinary integral of q if $q \in L^1(\mathbb{R})$ (see Theorem 4.5).

We also study the geometry of the Miura map. Whereas the Miura map on spaces of periodic functions is known to be a global fold, the situation for non-periodic functions is completely different. Let

$$E_1 = \left\{ q \in \mathrm{Im}(B) : B^{-1}(q) \text{ consists of a single point} \right\} \tag{1.5}$$

and

$$E_2 = \left\{ q \in \mathrm{Im}(B) : B^{-1}(q) \text{ is homeomorphic to an interval} \right\}. \tag{1.6}$$

Here we consider $B^{-1}(q)$ with the natural Fréchet topology of $L^2_{\mathrm{loc}}(\mathbb{R})$.

Theorem 1.3. $\mathrm{Im}(B) = E_1 \cup E_2$ *and both* E_1 *and* E_2 *are dense in* $\mathrm{Im}(B)$ *in the natural Fréchet topology of* $H^{-1}_{\mathrm{loc}}(\mathbb{R})$.

As an application of our results on the Miura map, we prove an existence result for solutions of the Korteweg-de Vries equation in $H^{-1}(\mathbb{R})$ for initial data in the range $\mathrm{Im}(B_0)$ of the Miura map $B_0 : L^2(\mathbb{R}) \to H^{-1}(\mathbb{R})$. We follow the approach of Tsutsumi [46], who proved such an existence result when the initial data is a positive, finite Radon measure on $\mathbb{R}$. His arguments, combined with our results on the Miura map B_0, lead to the following theorem.

Theorem 1.4. *Assume that* $u_0 \in \mathrm{Im}(B_0)$. *Then there exists a global weak solution of KdV with* $u(t) \in \mathrm{Im}\,(B_0)$ *for all* $t \in \mathbb{R}$. *More precisely:*
(i) $u \in L^\infty(\mathbb{R}, H^{-1}(\mathbb{R})) \cap L^2_{\mathrm{loc}}(\mathbb{R}^2)$;
(ii) for all functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, *the identity*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(-u\varphi_t - u\varphi_{xxx} + 3u^2 \varphi_x \right) \, dx \, dt = 0$$

holds, and
(iii) $\lim_{t\to 0} u(t) = u_0$ *in* $H^{-1}(\mathbb{R})$.

We state and prove a slightly stronger version of the above theorem in Section 6.

In Appendix C, we provide a few comments on the Miura map as well as on other work related to the results presented in this paper.

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2. Preliminaries

2.1. **Spaces of Distributions.** For $\alpha \in \mathbb{R}$, we denote by $H^\alpha(\mathbb{R})$ the completion of $\mathcal{C}_0^\infty(\mathbb{R})$ in the norm

$$\|\varphi\|_\alpha = \left(\int \left(1 + |\xi|^2 \right)^\alpha |\widehat{u}(\xi)|^2 \, \frac{d\xi}{2\pi} \right)^{1/2}$$

where

$$\widehat{u}(\xi) = \int \exp\left(-i\xi x\right) \, u(x) \, dx.$$

Clearly, $H^0(\mathbb{R}) = L^2(\mathbb{R})$ and $H^\alpha(\mathbb{R}) \subset H^\beta(\mathbb{R})$ if $\alpha \geq \beta$. Therefore $H^\alpha(\mathbb{R}) \subset L^2(\mathbb{R})$ if $\alpha \geq 0$. For any $\alpha \in \mathbb{R}$, $H^\alpha(\mathbb{R})$ is a space of tempered distributions. A distribution u belongs to $H^\alpha_{\mathrm{loc}}(\mathbb{R})$ if $\chi u \in H^\alpha(\mathbb{R})$ for any function $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$. We consider $H^\alpha_{\mathrm{loc}}(\mathbb{R})$ and $L^p_{\mathrm{loc}}(\mathbb{R})$ with $p \geq 1$ with their natural Fréchet topology - see [45], chapter 31-12.

A classical result of distribution theory (see, for example, chapter 1 of [14]) asserts that if $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$, then $q = Q'$ for a function $Q \in L^2_{\mathrm{loc}}(\mathbb{R})$. If $q \in H^{\beta-1}(\mathbb{R})$ for $\beta \geq 0$ we have a sharper result. Let

$$H^\infty(\mathbb{R}) = \bigcap_{\beta \geq 0} H^\beta(\mathbb{R}) \subset \mathcal{C}^\infty(\mathbb{R})$$

Lemma 2.1. *Let $\beta \geq 0$ and let $q \in H^{\beta-1}(\mathbb{R})$. There exist functions $f \in H^\beta(\mathbb{R})$ and $g \in H^\infty(\mathbb{R})$ so that $q = f' + g$ as elements of $H^{\beta-1}(\mathbb{R})$.*

Proof. Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\psi(\xi) = 1$ near $\xi = 0$, and choose

$$\widehat{f}(\xi) = (i\xi)^{-1} (1 - \psi(\xi)) \; \widehat{q}(\xi)$$

and

$$\widehat{g}(\xi) = \psi(\xi) \; \widehat{q}(\xi).$$

□

Finally, the following result will be useful in studying the continuity of the Miura map and the regularity of solutions to the Riccati equation $q = r' + r^2$.

Lemma 2.2. *The multiplication $\{u, v\} \mapsto uv$ can be extended from the bilinear map $\mathcal{C}_0^\infty(\mathbb{R}) \times \mathcal{C}_0^\infty(\mathbb{R}) \to \mathcal{C}_0^\infty(\mathbb{R})$ to continuous bilinear maps*

$$H^\beta(\mathbb{R}) \times H^\beta(\mathbb{R}) \to H^\beta(\mathbb{R}), \;\; \beta > 1/2, \tag{2.1}$$

$$H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \to H^{1/2-\delta}(\mathbb{R}) \textit{ for any } \delta > 0, \tag{2.2}$$

$$H^\beta(\mathbb{R}) \times H^\beta(\mathbb{R}) \to H^{2\beta-1/2}(\mathbb{R}), \;\; 0 < \beta < 1/2, \tag{2.3}$$

$$L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(\mathbb{R}) \subset H^{-1/2-\delta}(\mathbb{R}) \textit{ for any } \delta > 0. \tag{2.4}$$

Proof. All of the statements, except the last one, are particular cases of more general multidimensional results formulated, for example, in Theorem 1 of section 4.6.1 of [39]. The last statement is obvious except the last inclusion; this follows by duality from the imbedding $H^{1/2+\delta}(\mathbb{R}) \to L^\infty(\mathbb{R})$, which is a particular case of a well-known Sobolev imbedding theorem. □

Let us also introduce a notation $(\cdot, \cdot)$ for miscellaneous sesquilinear pairings extending the pairing

$$(u, v) = \int_{\mathbb{R}} u(x)\overline{v(x)} dx, \quad u, v \in \mathcal{C}_0^\infty(\mathbb{R}),$$

by continuity. In particular, we will use the extended pairings in the following cases:

(i) u is a distribution on $\mathbb{R}$, $v \in \mathcal{C}_0^\infty(\mathbb{R})$;

(ii) $u \in H^s(\mathbb{R})$, $v \in H^{-s}(\mathbb{R})$, where $s \in \mathbb{R}$;

(iii) $u \in H^s_{\mathrm{loc}}(\mathbb{R})$, $v \in H^{-s}_{comp}(\mathbb{R})$, where $s \in \mathbb{R}$ and $H^{-s}_{comp}(\mathbb{R})$ is the space of compactly supported distributions from $H^{-s}(\mathbb{R})$.

The integration by parts formula

$$(U', v) = -(U, v') \tag{2.5}$$

holds in all these cases, e.g. for $U \in H^{s+1}_{loc}(\mathbb{R})$, $v \in H^{-s}_{comp}(\mathbb{R})$, $s \in \mathbb{R}$, by continuity of the pairing.

2.2. **Continuity of the Miura Map.** It is easy to see that the Miura map defines a bounded continuous map from $L^2(\mathbb{R})$ into $H^{-1}(\mathbb{R})$ (we use the standard embedding $L^1(\mathbb{R}) \subset H^{-1}(\mathbb{R})$ which follows by duality from the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$). Localizing this, we see that the Miura map defines a bounded continuous mapping from $L^2_{\text{loc}}(\mathbb{R})$ into $H^{-1}_{\text{loc}}(\mathbb{R})$. This is extended to more general Sobolev spaces as follows.

Proposition 2.3. *Let $\beta \geq 0$. The Miura map is a continuous mapping from $H^\beta(\mathbb{R})$ into $H^{\beta-1}(\mathbb{R})$ and from $H^\beta_{\text{loc}}(\mathbb{R})$ into $H^{\beta-1}_{\text{loc}}(\mathbb{R})$ which is bounded, i.e. maps bounded subsets into bounded subsets.*

Remark 2.4. *For the definition of bounded sets in Fréchet spaces see e.g.* [45]*, Chapter 14, especially Proposition 14.5.*

Proof. It suffices to prove the first statement since the second follows by localization. It is clear that the map $r \mapsto r'$ is a bounded continuous map from $H^\beta(\mathbb{R})$ into $H^{\beta-1}(\mathbb{R})$ so it suffices to show that the map $r \mapsto r^2$ is a bounded continuous map from $H^\beta(\mathbb{R})$ into $H^{\beta-1}(\mathbb{R})$. This follows from Lemma 2.2 and the trivial embedding of $H^\alpha(\mathbb{R})$ into $H^\gamma(\mathbb{R})$ if $\alpha \geq \gamma$. $\square$

2.3. **A First-Order System.** Let $y \in H^1_{\text{loc}}(\mathbb{R})$ be a solution of the equation

$$-y'' + qy = 0 \tag{2.6}$$

where $q \in H^{-1}_{\text{loc}}(\mathbb{R})$. Then we can conveniently rewrite the equation in the form of a first-order system

$$\left\{ \begin{array}{rcl} y' & = & Qy + u \\ u' & = & -Q^2 y - Qu \end{array} \right. \tag{2.7}$$

where $Q \in L^2_{\text{loc}}(\mathbb{R})$ is a real-valued function with $Q' = q$ (this is a well-known procedure in the study of differential operators with singular coefficients; see §1.1 of [41] and references therein). This system is equivalent to equation (2.6) in the following sense. If $y \in H^1_{\text{loc}}(\mathbb{R})$ is a solution of (2.6), then taking $u = y' - Qy$ we obtain by straightforward substitution that the pair $\{y, u\}$ satisfies (2.7) in the sense of distributions. It follows that $u \in W^{1,1}_{\text{loc}}(\mathbb{R})$, the space of $L^1_{\text{loc}}(\mathbb{R})$-functions with distributional derivatives in $L^1_{\text{loc}}(\mathbb{R})$. On the other hand, if y and u belong to $W^{1,1}_{\text{loc}}(\mathbb{R})$ and the pair $\{y, u\}$ satisfies (2.7), then in fact $y \in H^1_{\text{loc}}(\mathbb{R})$ and y satisfies (2.6).

Since the coefficients of the linear system (2.7) are in $L^1_{\text{loc}}(\mathbb{R})$, the standard existence and uniqueness result holds for the corresponding initial value problem on the whole real line, and the solutions $\{y, u\}$ depend continuously on the initial data.

The next lemma shows that nonnegative solutions of $L_q y = 0$ are either strictly positive or identically zero.

Lemma 2.5. *Suppose that $y \in H^1_{\mathrm{loc}}(I)$ is a solution of $L_q y = 0$ with a real-valued $q \in H^{-1}_{\mathrm{loc}}(I)$, where I is an open interval in $\mathbb{R}$. Assume that $y(x_0) = 0$ for some $x_0 \in I$. Denote $u = y' - Qy$ as in (2.7). Then the following statements hold true:*

A. We have the following trichotomy for the behavior of y near x_0:

(i) If $u(x_0) = 0$, then $y \equiv 0$ on I.

(ii) If $u(x_0) > 0$ then in a neighborhood of x_0 we have $y(x) < 0$ for $x < x_0$, and $y(x_0) > 0$ for $x > x_0$.

(iii) If $u(x_0) < 0$ then in a neighborhood of x_0 we have $y(x) > 0$ for $x < x_0$, and $y(x_0) < 0$ for $x > x_0$.

B. If $y \not\equiv 0$, then all zeros of y on I are isolated.

C. If $y(x) \geq 0$ near x_0 then $y(x) \equiv 0$ on I.

Proof. Clearly, (i) follows from the uniqueness of solution $\{y, u\}$ of (2.7) with the given initial conditions $y(x_0) = y_0, u(x_0) = u_0$.

Define $z(x) = y(x) \exp\left(-\int_{x_0}^x Q(s)\, ds\right)$ and note that, as u is absolutely continuous, so is $z'(x) = u(x) \exp\left(-\int_{x_0}^x Q(s)\, ds\right)$. Clearly, $z(x_0) = 0$, and $z(x)$ has the same sign as $y(x)$ for all $x \in I$. Also, $u(x_0) > 0$ is equivalent to $z'(x_0) > 0$, so (ii) and (iii) immediately follow.

Clearly, B and C follow from A. □

3. Positivity, Positive Solutions, and the Miura Map

In this section we prove Theorem 1.1. First, we describe the connection between the Miura map and positive solutions of $L_q y = 0$. For a real-valued distribution $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$, let $\mathrm{Pos}(q)$ denote the (possibly empty) set of functions $y \in H^1_{\mathrm{loc}}(\mathbb{R})$ with the properties that $L_q y = 0$, $y(x) > 0$ for all $x \in \mathbb{R}$, and $y(0) = 1$.

Lemma 3.1. *Let q be a real-valued distribution belonging to $H^{-1}_{\mathrm{loc}}(\mathbb{R})$.*
(a) If $y \in \mathrm{Pos}(q)$ then $q = B(y'/y)$.
(b) If $q = B(r)$ for some $r \in L^2_{\mathrm{loc}}(\mathbb{R})$ then $y(x) = \exp\left(\int_0^x r(s)\, ds\right)$ belongs to $\mathrm{Pos}(q)$.

The proof is easily obtained by straightforward calculations.

The maps

$$\mathrm{Pos}(q) \ni y \mapsto \frac{d}{dx} \log(y(x)) \in B^{-1}(q)$$

and

$$B^{-1}(q) \ni r \mapsto \exp\left(\int_0^x r(s)\, ds\right) \in \mathrm{Pos}(q)$$

are continuous if we topologize $B^{-1}(q)$ with the topology induced from $L^2_{\mathrm{loc}}(\mathbb{R})$ and $\mathrm{Pos}(q)$ with the topology induced from $H^1_{\mathrm{loc}}(\mathbb{R})$. These maps are mutual inverses. Hence, we have shown:

Proposition 3.2. *The set $B^{-1}(q)$ is nonempty if and only if $\mathrm{Pos}(q)$ is nonempty. For any $r \in B^{-1}(q)$, $B^{-1}(B(r))$ is homeomorphic to $\mathrm{Pos}(B(r))$.*

Next, we show that if $L_q \geq 0$, then $\mathrm{Pos}(q)$ is nonempty. To this end, we introduce the sesquilinear forms

$$\mathfrak{t}_q(\varphi, \psi) = \int_{\mathbb{R}} \varphi'(x)\overline{\psi'}(x)dx + (q, \overline{\varphi}\psi) \tag{3.1}$$

and

$$\mathfrak{t}_{q,I}(\varphi,\psi) = \int_I \varphi'(x)\overline{\psi'}(x)\ dx + (q, \overline{\varphi}\psi) \tag{3.2}$$

defined respectively on $\mathcal{C}_0^\infty(\mathbb{R})$ and $\mathcal{C}_0^\infty(I)$, where $I = (a,b)$ is a bounded, open interval of $\mathbb{R}$.

The form $\mathfrak{t}_q$ is also well defined by (3.1) for $\varphi, \psi \in H^1_{comp}(\mathbb{R})$, where $H^1_{comp}(\mathbb{R})$ is the space of compactly supported functions from $H^1(\mathbb{R})$. Note that $\mathfrak{t}_q(\varphi,\varphi) = (L_q\varphi, \varphi)$ if $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, so that if $L_q \geq 0$, then both $\mathfrak{t}_q$ and $\mathfrak{t}_{q,I}$ are positive quadratic forms. Approximating $\varphi \in H^1_{comp}(\mathbb{R})$ by functions from $\mathcal{C}_0^\infty(\mathbb{R})$, we easily obtain that $\mathfrak{t}_q(\varphi,\varphi) \geq 0$ for all $\varphi \in H^1_{comp}(\mathbb{R})$ as well.

It is easy to see that $\mathfrak{t}_{q,I}$ admits a closure, which has the domain

$$H^1_0(I) = \left\{\psi \in H^1(I) : \psi(a) = \psi(b) = 0\right\}.$$

(See also Lemma 1.8 of [41].) It is a closed positive quadratic form which will also be denoted $\mathfrak{t}_{q,I}$. Note that if $\varphi, \psi \in H^1_0(I)$ and φ_0, ψ_0 are their extensions on $\mathbb{R}$ by 0, then $\varphi_0, \psi_0 \in H^1_{comp}(\mathbb{R})$, and

$$\mathfrak{t}_{q,I}(\varphi,\psi) = \mathfrak{t}_q(\varphi_0,\psi_0). \tag{3.3}$$

Let $L_{q,I}$ be the self-adjoint operator associated to $\mathfrak{t}_{q,I}$ by the Friedrichs construction. Clearly, $L_{q,I}$ has positive spectrum. Moreover, it has compact resolvent, or, equivalently, discrete spectrum. (This follows from the compactness of the imbedding of $H^1_0(I)$ into $L^2(I)$; see also [41], where the asymptotics of the eigenvalues is found.) By the min-max principle, the lowest eigenvalue of $L_{q,I}$ is given by

$$\lambda_0(I) = \inf\left\{\mathfrak{t}_{q,I}(\varphi,\varphi) : \varphi \in H^1_0(I) \text{ and } \|\varphi\|_{L^2(I)} = 1\right\} \geq 0$$

and the infimum is achieved by a corresponding eigenfunction $h \in H^1_0(I)$.

Lemma 3.3. *Let q be a real-valued distribution belonging to $H^{-1}_{\mathrm{loc}}(\mathbb{R})$. If $L_q \geq 0$, then $\lambda_0(I) > 0$ for every bounded open interval I.*

Proof. As $\lambda_0(I) \geq 0$ it suffices to show that no bounded interval I has $\lambda_0(I) = 0$. Suppose, on the contrary, that such an interval I exists, and let $h \in H^1_0(I)$ be an L^2-normalized, real-valued eigenfunction with the eigenvalue 0, so, in particular, $\mathfrak{t}_{q,I}(h,h) = 0$. Extend h to a function $\eta = h_0 \in H^1_{comp}(\mathbb{R})$ as above, i.e. by setting $\eta(x) = 0$ if $x \in \mathbb{R} \setminus I$. By (3.3), we conclude that $\mathfrak{t}_q(\eta,\eta) = \mathfrak{t}_{q,I}(h,h)$, hence $\mathfrak{t}_q(\eta,\eta) = 0$, So, for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $t \in \mathbb{R}$,

$$\mathfrak{t}_q(\eta + t\varphi, \eta + t\varphi) = 2t\ \operatorname{Re}\ \mathfrak{t}_q(\eta,\varphi) + t^2\ \mathfrak{t}_q(\varphi,\varphi)$$

is nonnegative due to positivity of $\mathfrak{t}_q$. It follows that $\operatorname{Re}\ \mathfrak{t}_q(\eta,\varphi) = 0$ for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, hence $\mathfrak{t}_q(\eta,\varphi) = 0$ also for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. It follows that η solves the equation $L_q\eta = 0$ and has a compact suppport. By Lemma 2.5 we obtain $\eta(x) = 0$ identically. This contradicts the assumption that $\|h\|_{L^2(I)} = 1$, and the lemma is proved. □

Corollary 3.4. *Let q be a real-valued distribution from $H^{-1}_{\mathrm{loc}}(\mathbb{R})$. If $L_q \geq 0$ and $y \in H^1_{\mathrm{loc}}(\mathbb{R})$ solves $L_q y = 0$, then y can have at most one zero.*

Proof. If $y(a) = y(b) = 0$ for $a < b$, then zero is a Dirichlet eigenvalue of $L_{q,I}$ with $I = (a,b)$, contradicting Lemma 3.3. □

Proposition 3.5. *Let q be a real-valued distribution which belongs to $H^{-1}_{\mathrm{loc}}(\mathbb{R})$. If $L_q \geq 0$, then $L_q y = 0$ has a strictly positive solution.*

Proof. For $c \in \mathbb{R}\setminus\{0\}$, let $\{y, u\} \in H^1_{\mathrm{loc}}(\mathbb{R}) \times W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ be the unique solution of (2.7) with $y(c) = 0$ and $u(c) = 1$. As y satisfies $L_q y = 0$, Corollary 3.4 implies that $y(0) \neq 0$. Denote by $\{y_c, u_c\}$ the scaled solution of (2.7),

$$y_c(x) = y(x)/y(0), \; u_c(x) = u(x)/y(0). \tag{3.4}$$

Then $y_c \in H^1_{\mathrm{loc}}(\mathbb{R})$ is the unique solution of $L_q y = 0$ with $y(0) = 1$ and $y(c) = 0$. Next, choose $c, c' \in \mathbb{R}$ so that $0 < c' < c$. Note that $w(x) = y_c(x) - y_{c'}(x)$ is a solution of $L_q y = 0$ with $w(0) = 0$, but $w(c) > 0$. By Corollary 3.4, $w(x) > 0$ for $x > 0$ and by Lemma 2.5 and Corollary 3.4 $w(x) < 0$ for $x < 0$. It follows that, on the half-line $c > 0$, the map $c \mapsto y_c(x)$ is monotone decreasing for any given $x < 0$ and monotone increasing for any given $x > 0$.

We wish to construct a positive solution of $L_q y = 0$ by taking the limit $c \to +\infty$. To this end denote by $(\tilde{y}_1, \tilde{u}_1)$ and $(\tilde{y}_2, \tilde{u}_2)$ the fundamental solutions of (2.7), determined by $\tilde{y}_1(0) = 1$, $\tilde{u}_1(0) = 0$ and $\tilde{y}_2(0) = 0, \tilde{u}_2(0) = 1$ respectively. By Lemma 3.3, $\tilde{y}_2(-1) \neq 0$. Hence, for any $\alpha \in \mathbb{R}$

$$z(x; \alpha) := \tilde{y}_1(x) + \frac{\alpha - \tilde{y}_1(-1)}{\tilde{y}_2(-1)} \tilde{y}_2(x)$$

is the unique solution of $L_q y = 0$ with $y(-1) = \alpha$ and $y(0) = 1$. Moreover, it follows that for any $x \in \mathbb{R}$, $\alpha \mapsto z(x; \alpha)$ is continuous. (In fact, the map $\alpha \mapsto z(\cdot; \alpha)$ is an affine, hence continuous map $\mathbb{R} \to H^1_{\mathrm{loc}}(\mathbb{R})$.)

Note that $y_c(x) = z(x; y_c(-1))$. Now consider the solution y_n of $L_q y = 0$, $y(0) = 1$, $y(n) = 0$. Then $\alpha_n = y_n(-1)$ is a strictly positive, decreasing sequence. Let $\alpha_\infty = \lim_{n\to\infty} \alpha_n$. By the continuity of $z(x; \alpha)$ with respect to α and the fact that $y_n(x) = z(x; \alpha_n)$, it then follows that $z(x; \alpha_\infty) = \lim_{n\to\infty} y_n(x)$ for any $x \in \mathbb{R}$. We claim that $\alpha_\infty > 0$ and that $z(x; \alpha_\infty)$ is positive. To prove this, first note that for any $n \geq 1$, $y_n(x) \geq 0$ for all $x \leq n$ and hence $\lim_{n\to\infty} y_n(x) \geq 0$ for all $x \in \mathbb{R}$. If $\alpha_\infty = 0$, then

$$0 \leq \lim_{n\to\infty} y_n(-2) = \lim_{n\to\infty} z(-2; \alpha_n) = z(-2; 0) = y_{-1}(-2) < 0,$$

a contradiction. Hence $\alpha_\infty > 0$. As a consequence, $z(x; \alpha_\infty)$ is a nonnegative solution of $L_q y = 0$ and hence strictly positive by Lemma 2.5. □

Proof of Theorem 1.1. In the statement of Theorem 1.1, we have (ii)⇒(i) by Lemma 3.1(a). To show that (i)⇒(iii), we compute that, for $q = r' + r^2$ and any $\varphi \in \mathcal{C}^\infty_0(\mathbb{R})$,

$$(L_q\varphi, \varphi) = \int |\varphi' - r\varphi|^2 \, dx \geq 0. \tag{3.5}$$

Finally, (iii)⇒(ii) by Proposition 3.5. □

4. The Image of the Miura Map

In this section we prove Theorem 1.2. Recall that B_β denotes the restriction of the Miura map to the Sobolev space $H^\beta(\mathbb{R})$ for $\beta \geq 0$.

We begin by considering the case $\beta = 0$. In the light of Theorem 1.1, we need to find necessary and sufficient conditions on a potential $q \in H^{-1}(\mathbb{R})$ so that there exists a solution $r \in L^2(\mathbb{R})$ of the Riccati equation $r' + r^2 = q$. Hartman [19], chapter

XI.7, Lemma 7.1 has studied this problem for continuous q and his arguments still apply in our more general setting.

Lemma 4.1. *Suppose that $q \in H^{-1}(\mathbb{R})$ is a real-valued distribution and that*

$$\sup_{|T|>1} \left| \frac{1}{T} \int_0^T Q(x)\ dx \right| < +\infty \tag{4.1}$$

for an antiderivative $Q \in L^2_{\text{loc}}(\mathbb{R})$ of q. Then every solution $r \in L^2_{\text{loc}}(\mathbb{R})$ of the Riccati equation $r' + r^2 = q$ belongs to $L^2(\mathbb{R})$. Conversely, if $r \in L^2(\mathbb{R})$, then every antiderivative Q of $q = r' + r^2$ satisfies (4.1).

Proof. (i) Assume that $Q \in L^2_{\text{loc}}(\mathbb{R})$, $Q' = q$, and Q satisfies (4.1). We need to show that for any solution $r \in L^2_{\text{loc}}(\mathbb{R})$ of $r' + r^2 = q$, the integrals $\int_0^\infty r^2(s)\ ds$ and $\int_{-\infty}^0 r^2(s)\ ds$ are finite. Let us show that the first integral is finite. Since Q is an antiderivative of q, it follows that, for a constant C,

$$r(x) + \int_0^x r^2(s)\ ds = Q(x) + C. \tag{4.2}$$

By assumption, the Césaro mean $T^{-1} \int_0^T (\ \cdot\ )\ dx$ of the right-hand side of (4.2) is bounded as $T \to +\infty$. We suppose that $\int_0^x r^2(s)\ ds \to +\infty$ as $x \to +\infty$ and obtain a contradiction as follows.

First note that if $f \in L^1_{\text{loc}}(\mathbb{R})$ and $f(x) \to +\infty$ as $x \to +\infty$ then the same holds for its Césaro mean, i.e.

$$\frac{1}{T} \int_0^T f(x)\ dx \to +\infty \qquad \text{as } T \to +\infty. \tag{4.3}$$

Hence, taking Césaro means of (4.2) we see that, if $\int_0^x r^2(s)\ ds \to +\infty$ as $x \to +\infty$, then $T^{-1} \int_0^T\ r(x)\ dx \to -\infty$ as $x \to +\infty$. Moreover, there is a T_0 so that for all $T > T_0$,

$$-\frac{2}{T} \int_0^T\ r(x)\ dx \geq \frac{1}{T} \int_0^T\ \left(\int_0^x\ r^2(s)\ ds \right)\ dx > 0.$$

By the Cauchy-Schwarz inequality,

$$-T^{-1} \int_0^T\ r(x)\ dx \leq T^{-1/2} \left(\int_0^T\ r^2(x)\ dx \right)^{1/2}$$

so that

$$4T \int_0^T\ r^2(x)\ dx \geq \left[\int_0^T \left(\int_0^x\ r^2(s)\ ds \right) dx \right]^2.$$

Setting $I(T) = \int_0^T \int_0^x\ r^2(s)\ ds\ dx$, we have that

$$4T\ I'(T) \geq I(T)^2$$

from which it follows by integration that

$$\frac{1}{I(T_0)} - \frac{1}{I(T)} \geq \frac{1}{4} \log\left(T/T_0\right).$$

This contradicts that, by (4.3), $I(T) \to +\infty$ as $T \to +\infty$. A similar argument shows that $\int_{-\infty}^0 r^2(s)\ ds$ is finite. Hence $r \in L^2(\mathbb{R})$.

(ii) If, on the other hand, $q = r' + r^2$ for $r \in L^2(\mathbb{R})$, then the function $Q(x) = r(x) + \int_0^x r^2(s)\, ds$ satisfies (4.1) by the Cauchy-Schwarz inequality applied in $[0, T]$. □

Corollary 4.2. *Suppose that $q \in \mathrm{Im}(B_0)$, i.e., $q = r' + r^2$ for some $r \in L^2(\mathbb{R})$. If $u \in L^2_{\mathrm{loc}}(\mathbb{R})$ solves the Riccati equation $u' + u^2 = q$, then $u \in L^2(\mathbb{R})$.*

Proposition 4.3. *A real-valued distribution $q \in H^{-1}(\mathbb{R})$ belongs to $\mathrm{Im}(B_0)$ if and only if*
(i) $L_q \geq 0$, and
(ii) q can be presented as $q = f' + g$ for real-valued functions $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$.

Proof. Suppose that $q \in \mathrm{Im}(B_0)$, i.e., $q = r' + r^2$ for some $r \in L^2(\mathbb{R})$. Then $L_q \geq 0$ by Theorem 1.1 and $q = f' + g$ with $f = r$, $g = r^2$. On the other hand, suppose that $q \in H^{-1}(\mathbb{R})$ with $L_q \geq 0$, and $q = f' + g$ for $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. By Theorem 1.1, $q \in \mathrm{Im}(B)$, so $q = r' + r^2$ for some $r \in L^2_{\mathrm{loc}}(\mathbb{R})$. The antiderivative $Q(x) = f(x) + \int_0^x g(s)\, ds$ obeys the condition (4.1), so $r \in L^2(\mathbb{R})$ by Lemma 4.1. □

Proposition 4.4. *The set $\mathrm{Im}(B_0)$ has no interior points, and hence is not open in $H^{-1}(\mathbb{R})$. Further, the set $\mathrm{Im}(B_0)$ is not closed in $H^{-1}(\mathbb{R})$.*

Proof. First we show that $\mathrm{Im}(B_0)$ has empty interior. If $q \in \mathcal{C}_0^\infty(\mathbb{R}) \cap \mathrm{Im}(B_0)$, we can perturb q by a small potential well far separated from the support of q and create a bound state. More precisely, for $\varepsilon > 0$, let

$$v_\varepsilon(x) = \begin{cases} -\varepsilon & |x| < 1/(2\varepsilon) \\ 0 & |x| \geq 1/(2\varepsilon) \end{cases}.$$

Observe that $\int v_\varepsilon(x) dx = -1$ but $\|v_\epsilon\|_{L^2(\mathbb{R})} = \varepsilon$. Suppose that $q \in \mathcal{C}_0^\infty(\mathbb{R}) \cap \mathrm{Im}(B_0)$ with support contained in $[-a, a]$. Let $w_\varepsilon(x) = v_\varepsilon(x - 2a - 2\varepsilon^{-1})$; then w_ε has support disjoint from the one of q. By choosing ε sufficiently small, we can assure that the potential $q_\varepsilon = q + w_\varepsilon$ is close to q in $L^2(\mathbb{R})$ norm. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ be a nonnegative function with $\chi(x) = 1$ for $|x| < 1/(2\varepsilon)$, $\chi(x) = 0$ for $|x| > \varepsilon^{-1}$, and $|\chi'(x)| \leq 3\varepsilon$. Finally, let $\eta(x) = \chi\left(x - 2a - 2\varepsilon^{-1}\right)$. Then

$$(L_{q_\varepsilon}\eta, \eta) = \int |\eta'(x)|^2 - 1 \leq 18\varepsilon - 1.$$

Hence, by Theorem 1.1, $q_\varepsilon \notin \mathrm{Im}(B)$ for $0 < \varepsilon < 1/18$. Since $\mathcal{C}_0^\infty(\mathbb{R})$ is norm-dense in $H^{-1}(\mathbb{R})$, this shows that $\mathrm{Im}(B_0)$ contains no open neighborhood in the norm topology of $H^{-1}(\mathbb{R})$.

Next, we show that $\mathrm{Im}(B_0)$ is not closed. Suppose that q is any nonnegative function with $q \in L^2(\mathbb{R})$ but $q \notin L^1(\mathbb{R})$. We may approximate q by nonnegative potentials $q_k \in \mathcal{C}_0^\infty(\mathbb{R})$ so that $q_k \to q$ in $L^2(\mathbb{R})$. Hence, $q_k \to q$ in $H^{-1}(\mathbb{R})$ and by Proposition 4.3, $q_k \in \mathrm{Im}(B_0)$ for any $k \geq 1$. Moreover, since $L_q \geq 0$, it follows from Theorem 1.1 that $q \in \mathrm{Im}(B)$. On the other hand, as $q \geq 0$, no antiderivative Q of q satisfies condition (4.1). Thus $\mathrm{Im}(B_0)$ is not closed in the norm topology on $H^{-1}(\mathbb{R})$. □

The image of B_0 can also be characterized by a ‘special integral’ of q. Let $\{\chi_n\}_{n\geq 1}$ be a sequence of nonnegative $\mathcal{C}_0^\infty(\mathbb{R})$ functions with (i) $\chi_n(x) = 1$ for $|x| \leq n$, (ii) $\chi_n(x) = 0$ for $|x| \geq n + 1$, and (iii) $|\chi_n'(x)| \leq 2$ for all $x \in \mathbb{R}$. Given $q \in H^{-1}(\mathbb{R})$, we define the special integral of q, denoted $[q]$, to be the number $\lim_{n\to\infty}(q,\chi_n)$ if this limit exists and is finite. One easily checks that $[q]$ is well-defined, i.e., does not depend on the choice of sequence $\{\chi_n\}_{n\geq 1}$ satisfying properties (i), (ii), and (iii) above. If $q \in \mathrm{Im}(B_0)$ then, for any $r \in B_0^{-1}(q)$,

$$\lim_{n\to\infty}(q,\chi_n) = \lim_{n\to\infty}\left\{(-r,\chi_n') + \left(r^2,\chi_n\right)\right\} = \|r\|^2_{L^2(\mathbb{R})} \tag{4.4}$$

which shows that $[q] \geq 0$ on $\mathrm{Im}(B_0)$ with $[q] = 0$ if and only if $q = 0$. Moreover, if r_1 and r_2 belong to $B^{-1}(q)$, then $\|r_1\|_{L^2(\mathbb{R})} = \|r_2\|_{L^2(\mathbb{R})}$.

The special integral has the following properties:

(a) $[f] = \int_{\mathbb{R}} f\ dx$ if $f \in L^1(\mathbb{R}) \subset H^{-1}(\mathbb{R})$;
(b) $\mathrm{Dom}\left([\ \cdot\ ]\right)$ is a linear subspace in $H^{-1}(\mathbb{R})$, and $f \mapsto [f]$ is linear;
(c) $[f'] = 0$ for any $f \in L^2(\mathbb{R})$;
(d) If $f \in L^2(\mathbb{R})$, then $[f]$ exists if and only if f is conditionally integrable, i.e. the limit $\lim_{T\to\infty}\int_{-T}^{T} f(x)\ dx$ exists. In this case, the limit equals $[f]$.

Using the special integral we can give an alternative characterization of $\mathrm{Im}(B_0)$.

Theorem 4.5. *A real-valued distribution $q \in H^{-1}(\mathbb{R})$ belongs to $\mathrm{Im}(B_0)$ if and only if:*
(i) $L_q \geq 0$, and
(ii) $[q]$ exists.
Moreover, for any $q \in \mathrm{Im}(B_0)$, one has $[q] \geq 0$.

Proof. First, suppose that $L_q \geq 0$ and $[q]$ exists. To prove that $q \in \mathrm{Im}(B_0)$, it suffices by Lemma 4.1 to show that q has an antiderivative Q with bounded Césaro means. By Lemma 2.1, any $q \in H^{-1}(\mathbb{R})$ may be written $q = f' + g$ for f and g belonging to $L^2(\mathbb{R})$. We can therefore take $Q(x) = f(x) + G(x)$ where $G(x) = \int_0^x g(s)\ ds$. We will use condition (ii) on q to show that G is bounded. Since $[f'] = 0$ for any $f \in L^2(\mathbb{R})$, we have $[q] = [g]$ and $[g]$ exists. Since, also, $g \in L^2(\mathbb{R})$, the existence of $[g]$ implies that g is conditionally integrable. Thus, $\lim_{n\to\infty}\alpha_n$ exists where $\alpha_n := \int_{-n}^{n} g(x)\ dx$. This is equivalent to the existence of $\lim_{a\to+\infty}\int_{-a}^{a} g(x)\ dx$ if $g \in L^2(\mathbb{R})$. We need to show that the numbers $\alpha_n^+ = \int_0^n\ g(x)\ dx$ and $\alpha_n^- = \int_{-n}^0\ g(x)\ dx$ are also bounded. Let $\{\eta_n\}_{n\geq 1}$ be a sequence of $\mathcal{C}_0^\infty(\mathbb{R})$ functions with $0 \leq \eta_n(x) \leq 1$, $\eta_n(x) = 1$ for $x \in [0,n]$, $\eta_n(x) = 0$ for $x \in \mathbb{R}\backslash[-1,n+1]$, and $|\eta_n'(x)| \leq 2$. Since $(L_q\eta_n,\eta_n) \geq 0$,

$$-\left(f, 2\eta_n\eta_n'\right) + \int g\eta_n^2\ dx \geq -\left\|\eta_n'\right\|^2_{L^2(\mathbb{R})}.$$

Since $\|\eta_n'\|_{L^2(\mathbb{R})} \leq 4$ and

$$|(f, 2\eta_n\eta_n')| \leq 4\int_{-1}^0 |f(x)|\ dx + 4\int_n^{n+1} |f(x)|\ dx \leq 8\,\|f\|_{L^2(\mathbb{R})}$$

as well as

$$\int g\eta_n^2\ dx = \alpha_n^+ + \int_{-1}^0 g(x)\eta_n^2(x)\ dx + \int_n^{n+1} g(x)\eta_n^2(x)\ dx \leq \alpha_n^+ + 2\,\|g\|_{L^2(\mathbb{R})}$$

it then follows that $\alpha_n^+ \geq -C$ with C independent of n. A similar argument shows that $\alpha_n^- \geq -C$ with C independent of n. As $\alpha_n = \alpha_n^+ + \alpha_n^-$ we conclude that the sequences $\{\alpha_n^+\}$ and $\{\alpha_n^-\}$ are both bounded, so G is bounded. Since $L_q \geq 0$ we have $r' + r^2 = q$ for some $r \in L^2_{\mathrm{loc}}(\mathbb{R})$ by Theorem 1.1, and applying Lemma 4.1 we conclude that $r \in L^2(\mathbb{R})$. Hence $q \in \mathrm{Im}(B_0)$.

On the other hand, if $q \in \mathrm{Im}(B_0)$, then $L_q \geq 0$ by Theorem 1.1 and $q = r' + r^2$ for some $r \in L^2(\mathbb{R})$, hence by properties (c) and (d) of the special integral, $[q]$ exists and $[q] = \|r\|^2_{L^2(\mathbb{R})} \geq 0$. □

Corollary 4.6. *An odd distribution $q \in H^{-1}(\mathbb{R})$ cannot be in $\mathrm{Im}(B_0)$ unless $q \equiv 0$.*

Remark 4.7. *Generally, the condition $L_q \geq 0$ can be considered as a weak form of positivity for q. If it is satisfied then the existence of the special integral $[q]$ for $q \in H^{-1}(\mathbb{R})$ implies much stronger existence-of-limit type results. For example, let us take any family of functions $\chi_{T_1,T_2} \in \mathcal{C}_0^\infty(\mathbb{R})$, $T_1, T_2 \in \mathbb{R}$, such that $\chi_{T_1,T_2} = 1$ on $[-T_1, T_2]$, $\chi_{T_1,T_2} = 0$ on $\mathbb{R} \setminus (-T_1 - 1, T_2 + 1)$, and the derivatives $\chi'_{T_1,T_2}(x)$ are uniformly bounded. Then $L_q \geq 0$ and existence of $[q]$ imply the existence of the limit*

$$\lim_{T_1,T_2 \to +\infty} \langle q, \chi_{T_1,T_2} \rangle,$$

which in case $q \in L^2(\mathbb{R})$ is equivalent to the existence of the limit

$$\lim_{T_1,T_2 \to +\infty} \int_{-T_1}^{T_2} q(x)dx.$$

To prove the above statements we can, for example, use Theorem 4.5 to find $r \in L^2(\mathbb{R})$, such that $q = r' + r^2$, and the result easily follows.

We now consider the restriction $B_\beta : H^\beta(\mathbb{R}) \to H^{\beta-1}(\mathbb{R})$ for $\beta > 0$.

Lemma 4.8. *Let $\beta \geq 0$. If $q \in \mathrm{Im}(B_0) \cap H^{\beta-1}(\mathbb{R})$ and $r \in L^2_{\mathrm{loc}}(\mathbb{R})$ is a solution of the Riccati equation $r' + r^2 = q$, then $r \in H^\beta(\mathbb{R})$.*

Proof. By Corollary 4.2, the result holds for $\beta = 0$. Hence, it suffices to prove that in case the claimed result holds for a given $\beta_0 \geq 0$, it also holds for any $\beta \in [\beta_0, \beta_0 + 1/4]$. So, assume that $q := r' + r^2 \in H^{\beta-1}(\mathbb{R})$ with $\beta_0 < \beta \leq \beta_0 + \frac{1}{4}$ and $r \in H^{\beta_0}(\mathbb{R})$. Then r^2 belongs to $H^{-1/2-\delta}(\mathbb{R})$, $H^{2\beta_0 - 1/2}(\mathbb{R})$, $H^{1/2-\delta}(\mathbb{R})$, or $H^{\beta_0}(\mathbb{R})$ respectively when $\beta_0 = 0$, $0 < \beta_0 < 1/2$, $\beta_0 = 1/2$, or $\beta_0 > 1/2$ (see (2.1)-(2.4)). In the first and last cases, $\delta > 0$ can be chosen arbitrarily small. Then $r' = q - r^2$ is in $H^{s-1}(\mathbb{R})$ with $s = \min(\beta, 1/2-\delta)$, $\min(\beta, 2\beta_0 + 1/2)$, $\min(\beta, 3/2-\delta)$ or $\min(\beta, \beta_0 + 1)$ respectively. As $r \in L^2(\mathbb{R})$ it then follows that $r \in H^s(\mathbb{R})$ and since $\beta_0 \leq \beta \leq \beta_0 + 1/4$ implies $s = \beta$ in all cases, we get the desired result. □

Proof of Theorem 1.2. First, suppose that $q \in H^{\beta-1}(\mathbb{R})$ satisfies conditions (i) and (ii) of Theorem 1.2. From the trivial inclusion $H^\beta(\mathbb{R}) \subset H^0(\mathbb{R})$ and Proposition 4.3, it follows that $q \in \mathrm{Im}(B_0)$, i.e., $q = r' + r^2$ for some function $r \in L^2(\mathbb{R})$. Applying Lemma 4.8 we see that $r \in H^\beta(\mathbb{R})$, so $q \in \mathrm{Im}(B_\beta)$ as claimed. Second, if $q \in \mathrm{Im}(B_\beta)$, then $L_q \geq 0$ by Theorem 1.1 and $q = f' + g$ with $f = r \in L^2(\mathbb{R})$ and $g = r^2 \in L^1(\mathbb{R})$. □

5. Geometry of the Miura Map

In this section, we prove Theorem 1.3. According to Proposition 3.2, $B^{-1}(q)$ is homeomorphic to the set of positive solutions y of $L_q y = 0$ with $y \in H^1_{\mathrm{loc}}(\mathbb{R})$ and $y(0) = 1$. As before we denote this set by $\mathrm{Pos}(q)$. We will show that $\mathrm{Pos}(q)$ is either a point or homeomorphic to a line segment.

Suppose that $\mathrm{Pos}(q)$ is nonempty and choose $y_1 \in \mathrm{Pos}(q)$. Using the Wronskian we can find another solution

$$y_2(x) = y_1(x) \int_0^x y_1(s)^{-2} \, ds.$$

The general solution to $L_q y = 0$ is then

$$y(x) = y_1(x) \left(c_1 + c_2 F(x)\right)$$

where

$$F(x) = \int_0^x y_1(s)^{-2} ds.$$

Observe that F is a monotone increasing function with $F(0) = 0$. If we define numbers $m_\pm \in (0, +\infty]$ by

$$m_+ = \lim_{x \to +\infty} F(x) \tag{5.1}$$

and

$$m_- = -\lim_{x \to -\infty} F(x), \tag{5.2}$$

then F takes values in $(-m_-, m_+)$. We will set $m_+^{-1} = 0$ if $m_+ = +\infty$, and similarly for m_-^{-1}. The conditions $y(0) = 1$ and $y(x) > 0$ for all x determine that any $y \in \mathrm{Pos}(q)$ is written

$$y(x) = y_1(x) \left(1 + cF(x)\right)$$

with $c \in \left[-m_+^{-1}, m_-^{-1}\right]$. Letting

$$y_+(x) = y_1(x) \left(1 - m_+^{-1} F(x)\right) \tag{5.3}$$

$$y_-(x) = y_1(x) \left(1 + m_-^{-1} F(x)\right) \tag{5.4}$$

we see that

$$y_+(x) \le y(x) \le y_-(x), \;\; x > 0 \tag{5.5}$$

$$y_-(x) \le y(x) \le y_+(x), \;\; x < 0 \tag{5.6}$$

for any $y \in \mathrm{Pos}(q)$, and

$$\mathrm{Pos}(q) = \left\{ \theta y_+ + (1-\theta) y_- : \theta \in [0,1] \right\}.$$

Thus, either (i) $m_+ = m_- = +\infty$, $y_+ = y_-$ and $\mathrm{Pos}(q)$ consists of a single element, or (ii) at least one of $m_\pm$ is finite, $y_+ \ne y_-$. Noting that $\theta \mapsto \theta y_+ + (1-\theta) y_-$ is a continuous map from $[0,1]$ to the Hausdorff space $H^1_{\mathrm{loc}}(\mathbb{R})$ we see that $\mathrm{Pos}(q)$ is homeomorphic to the interval $[0,1]$. We have proved:

Lemma 5.1. *Suppose that $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$ is a real-valued distribution and $L_q \ge 0$. Then $\mathrm{Pos}(q)$ is either a point or homeomorphic to a line segment.*

Next, we show that the sets E_1 and E_2 defined in (1.5) and (1.6) are both dense in $H^{-1}_{\mathrm{loc}}(\mathbb{R})$. We begin with a simple lemma which will be useful in the proof of Theorem 1.3.

Lemma 5.2. *There exists a family of potentials $\{w_\varepsilon\}_{\varepsilon>0}$ contained in $\mathcal{C}_0^\infty(\mathbb{R}) \cap E_2$ so that (i) $\mathrm{supp}(w_\varepsilon) \subset \left[-\varepsilon^{-1}, \varepsilon^{-1}\right]$ and (ii) $\|w_\varepsilon\|_{H^\beta(\mathbb{R})} \to 0$ as $\varepsilon \downarrow 0$ for any $\beta \in \mathbb{R}$.*

Proof. Let $y \in \mathcal{C}^\infty(\mathbb{R})$ with $y(x) = 1$ for $x < -1$, $y(x) = x$ for $x > 1$, and $y(x) > 0$ for any $x \in \mathbb{R}$. The potential $w(x) = y''(x)/y(x)$ has y as a positive solution of $L_w y = 0$ and $w = B(r)$ with $r = y'/y$. Since $\int_0^\infty y(s)^{-2} ds < \infty$, it follows from the remarks preceding Lemma 5.1 that $w \in E_2$. Now let $y_\varepsilon(x) = y(\varepsilon x)$ and $w_\varepsilon(x) = y_\varepsilon''(x)/y_\varepsilon(x)$. Then $w_\varepsilon \in E_2$ with support in $\left[-\varepsilon^{-1}, \varepsilon^{-1}\right]$, proving (i). To prove (ii), note that $w_\varepsilon(x) = \varepsilon^2 w(\varepsilon x)$ so that for any nonnegative integer j,

$$\left\|\partial_x^j w_\varepsilon\right\|_{L^2(\mathbb{R})}^2 = \varepsilon^{3+2j} \left\|\partial_x^j w\right\|_{L^2(\mathbb{R})}^2 .$$

Since $\|u\|_{H^\alpha(\mathbb{R})} \leq \|u\|_{H^\beta(\mathbb{R})}$ for $\alpha < \beta$ and $u \in H^\beta(\mathbb{R})$, this shows that $\|w_\varepsilon\|_{H^\beta(\mathbb{R})} \to 0$ as $\varepsilon \downarrow 0$, for any $\beta \in \mathbb{R}$. □

Lemma 5.3. *$B(\mathcal{C}_0^\infty(\mathbb{R}))$ is dense in* $\mathrm{Im}(B)$ *and $B(\mathcal{C}_0^\infty(\mathbb{R})) \subset E_1$.*

Proof. Let $q \in \mathrm{Im}(B)$ and let $r \in B^{-1}(q)$. Let $\{r_n\} \in \mathcal{C}_0^\infty(\mathbb{R})$ with $r_n \to r$ in $L^2_{\text{loc}}(\mathbb{R})$. By the continuity of the Miura map, $B(r_n) \to B(r)$ in $H^{-1}_{\text{loc}}(\mathbb{R})$. Thus $B\left(\mathcal{C}_0^\infty(\mathbb{R})\right)$ is dense in $\mathrm{Im}(B)$. If $q = B(r)$ for $r \in \mathcal{C}_0^\infty(\mathbb{R})$, then $\mathrm{Pos}(q)$ contains the element $y_1(x) = \exp\left(\int_0^x r(s)\, ds\right)$ which is bounded above and below by strictly positive constants. It follows that $m_+ = m_- = +\infty$ (see (5.1) and (5.2)), By the analysis of positive solutions preceding Lemma 5.1, $y_+ = y_-$ and $\mathrm{Pos}(q)$ consists of a single point. Hence $B(\mathcal{C}_0^\infty(\mathbb{R})) \subset E_1$. □

On the other hand:

Lemma 5.4. *E_2 is dense in* $\mathrm{Im}(B)$.

Proof. Since $B(\mathcal{C}_0^\infty(\mathbb{R}))$ is dense in $\mathrm{Im}(B)$, it suffices to show that for any $q \in B(\mathcal{C}_0^\infty(\mathbb{R}))$ there is a sequence of elements q_n from E_2 with $q_n \to q$ in $H^{-1}_{\text{loc}}(\mathbb{R})$ as $n \to \infty$. Suppose that $q \in B(\mathcal{C}_0^\infty(\mathbb{R}))$ with support in $[-a, a]$ for $a > 0$ and consider the sequence

$$q_n = q + v_n$$

for $n \geq 1$, where

$$v_n(x) = w_{1/n}(x - a - 2n)$$

and w_ε is the family constructed in Lemma 5.2. Then $v_n \to 0$ for $n \to \infty$ in $H^\beta(\mathbb{R})$ for any $\beta \in \mathbb{R}$. Let $p \in H^1_{\text{loc}}(\mathbb{R})$ be the unique positive solution to $L_q p = 0$ with $p(0) = 1$; note that $p(x)$ is constant away from the support of q. If y_ε is the positive solution for w_ε constructed in Lemma 5.2, it is easily seen that the function

$$z_n(x) = \begin{cases} p(x), & x < a+1, \\ p(a+1) y_{1/n}(x - a - 2n), & x \geq a+1, \end{cases} \tag{5.7}$$

is a positive solution to $L_{q_n} y = 0$. It follows from (5.7) and the fact that $y_{1/n}(x) = x/n$ for x large and positive that $\int_0^\infty z_n(s)^{-2} ds < \infty$. Thus, by the analysis of positive solutions preceding Lemma 5.1, $q_n \in E_2$. Since $v_n \to 0$ in $H^\beta(\mathbb{R})$ for any $\beta \in \mathbb{R}$, $q_n - q \to 0$ in $H^{-1}_{\text{loc}}(\mathbb{R})$. □

Proof of Theorem 1.3. That $\mathrm{Im}(B) = E_1 \cup E_2$ follows from Lemma 5.1 and Proposition 3.2. The density statements were proved in Lemmas 5.3 and 5.4. □

We close this section with some further remarks on the dichotomy of the Miura map. First, we give a version of Theorem 1.3 for the restriction of the Miura map to $H^\beta(\mathbb{R})$, $\beta \geq 0$.

Theorem 5.5. *$\operatorname{Im}(B_\beta) = E_{1,\beta} \cup E_{2,\beta}$ where $E_{j,\beta} = E_j \cap \operatorname{Im}(B_0) \cap H^{\beta-1}(\mathbb{R})$, $j = 1, 2$. Moreover $E_{j,\beta}$ is dense in $\operatorname{Im}(B_\beta)$ for $j = 1, 2$.*

Proof. The first statement follows from the fact, established in Theorem 1.2, that $\operatorname{Im}(B_\beta) = \operatorname{Im}(B_0) \cap H^{\beta-1}(\mathbb{R})$. The proofs of Lemmas 5.3 and 5.4 can be adapted with trivial changes to show the density of $E_{j,\beta}$ in $\operatorname{Im}(B_\beta)$. □

Finally, let

$$\lambda_0(q) = \inf\left\{(L_q\varphi, \varphi) : \varphi \in \mathcal{C}_0^\infty(\mathbb{R}),\ \|\varphi\|_{L^2(\mathbb{R})} = 1\right\}, \tag{5.8}$$

or, equivalently,

$$\lambda_0(q) = \inf\left\{\frac{\mathsf{t}_q(\psi,\psi)}{(\psi,\psi)} : \psi \in H^1_{\mathrm{comp}}(\mathbb{R}) \setminus \{0\}\right\}, \tag{5.9}$$

and define the sets

$$E_\bullet = \{q \in \operatorname{Im}(B) : \lambda_0(q) = 0\}$$

and

$$E_> = \{q \in \operatorname{Im}(B) : \lambda_0(q) > 0\}.$$

If q has compact support, it is clear that $\lambda_0(q) = 0$ since we can choose test functions whose support is disjoint from the support of q and

$$\inf\left\{\|\varphi'\|^2 : \varphi \in \mathcal{C}_0^\infty(\mathbb{R}),\ \|\varphi\|_{L^2(\mathbb{R})} = 1\right\} = 0.$$

Note that the map $E_\bullet \times \mathbb{R}_{>0} \to E_>$ given by

$$(q, c) \mapsto q + c$$

is a continuous, bijective map onto $E_>$.

Theorem 5.6. *(i) $E_> \subset E_2$ and $E_1 \subseteq E_\bullet$.*
(ii) $E_2 \cap E_\bullet \neq \emptyset$, i.e., $E_\bullet$ is not a fold of the dichotomy. Moreover, $E_\bullet$ is dense in $\operatorname{Im}(B)$, and E_1 and $E_2 \cap E_\bullet$ are dense in $E_\bullet$.

Remark 5.7. *The fact that $E_2 \cap E_\bullet \neq \emptyset$ has already been observed by Murata* [36], *Remark to Theorem 2.2, in his investigation of critical and subcritical potentials – see Appendix C.*

The proof of Theorem 5.6 will rely on the following proposition.

Proposition 5.8. *Suppose that $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$ and $\lambda_0(q) > 0$. Then the equation $L_q y = 0$ has two linearly independent positive solutions $y_1, y_2 \in H^1_{\mathrm{loc}}(\mathbb{R})$.*

Remark 5.9. *For potentials $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, the result above is due to Murata* [36], *Remark after Theorem 2.7.*

In the proof of Proposition 5.8 we will use

Lemma 5.10. *Assume that $y \in H^1_{\mathrm{loc}}(\mathbb{R})$, and there exists a discrete subset $S \subset \mathbb{R}$, such that $L_q y = 0$ on $\mathbb{R} \setminus S$. Then*

$$L_q y = \sum_{z \in S} (u(z-0) - u(z+0))\delta(\cdot - z), \tag{5.10}$$

where $u = y' - Qy$, $Q' = q$ as in (2.7). In other words,

$$(y, L_q\varphi) = \sum_{z \in S} (u(z-0) - u(z+0))\overline{\varphi(z)}, \tag{5.11}$$

for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$.

Proof. Using partition of unity, we can split any function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ into a finite sum of functions φ_k such that for every k a neighborhood of $\operatorname{supp}\varphi_k$ contains at most one point from S. Therefore, taking into account translation invariance, we see that it suffices to consider the case when $S = \{0\}$. So we will assume that $y \in H^1_{\mathrm{loc}}(\mathbb{R})$ and $L_q y = 0$ on $\mathbb{R} \setminus 0$.

Integrating by parts (see (2.5)) and using (2.7), we get

$$\begin{aligned}(y, L_q\varphi) &= (y', \varphi') + (qy, \varphi)\\ &= (u + Qy, \varphi') + (qy, \varphi) = (u, \varphi') + (qy - (Qy)', \varphi)\\ &= (u, \varphi') - (Qy', \varphi) = (u, \varphi') - (Q^2 y + Qu, \varphi).\end{aligned}$$

Integrating by parts in the first term in the right hand side we obtain

$$(u, \varphi') = \int_{-\infty}^0 u\overline{\varphi'}dx + \int_0^\infty u\overline{\varphi'}dx = (u(-0) - u(+0))\overline{\varphi(0)} - ([u'], \varphi),$$

where $[u']$ is the locally integrable function on $\mathbb{R}$ which coincides with u' on $\mathbb{R}\setminus\{0\}$. Since $[u'] = -Q^2 y - Qu$ due to (2.7), we finally obtain

$$(y, L_q\varphi) = (u(-0) - u(+0))\overline{\varphi(0)},$$

as required. □

Corollary 5.11. *Let y satisfy the conditions of Lemma 5.10 and have a compact support (so that S can be taken finite). Then*

$$\mathfrak{t}_q(y, y) = \int_{\mathbb{R}} (|y'|^2 + q|y|^2)dx = \sum_{z\in S} (u(z - 0) - u(z + 0))\overline{y(z)}. \tag{5.12}$$

Proof. Taking limit in (5.11) over a sequence φ_k converging to y in $H^1_{\mathrm{comp}}(\mathbb{R})$, we obtain (5.12). □

Proof of Proposition 5.8. Using notations from the proof of Proposition 3.5 (see (3.4)), for any $c > 0$ define a test function (to use in (5.9))

$$\psi_c(x) = \begin{cases} y_{-c}(x), & x \in (-c, 0);\\ y_c(x), & x \in [0, c);\\ 0, & x \notin (-c, c).\end{cases} \tag{5.13}$$

Clearly, $\psi_c \in H^1_{\mathrm{comp}}(\mathbb{R})$, $\psi_c(0) = 1$, and $L_q\psi_c = 0$ on $\mathbb{R} \setminus S$ where $S = \{-c, 0, c\}$.

Applying Corollary 3.4 to $y_c - y_{c'}$, we see that $c \mapsto \psi_c(x)$ is an increasing function of $c > 0$ for any fixed $x \in \mathbb{R}$. Therefore, the L^2-norm $\|\psi_c\|$ increases with c as well.

By Corollary 5.11 we have

$$\mathfrak{t}_q(\psi_c, \psi_c) = u_{-c}(0) - u_c(0).$$

It follows from Lemma 2.5 that $u_{-c}(0)$ decreases and $u_c(0)$ increases as c increases. Therefore, $c \mapsto \mathfrak{t}_q(\psi_c, \psi_c)$ is decreasing as c increases. It follows that the fraction in (5.9) is decreasing as well. To prove the desired statement, it is enough to establish that the limit of this fraction is 0 as $c \to +\infty$, provided we know that the equation $L_q y = 0$ has only one positive solution with $y(0) = 1$. To this end note that the limits of y_{-c} and y_c exists and are both positive solutions, according to the arguments given in the proof of Proposition 3.5. Due to our uniqueness of positive solution assumption these limits should coincide. But then we should also have

$$\lim_{c\to+\infty} u_{-c}(0) = \lim_{c\to+\infty} u_c(0),$$

because $y_{-c}(0) = y_c(0) = 1$ and the map $y \mapsto \{y(0), u(0)\}$ is a linear topological isomorphism between the space of all solutions of $L_q y = 0$ with the $H^1_{\text{loc}}(\mathbb{R})$-topology and the space $\mathbb{C}^2$ It follows that

$$\lim_{c \to +\infty} \mathfrak{t}_q(\psi_c, \psi_c) = 0,$$

which implies the desired statement. □

Proof of Theorem 5.6. To prove part (i), it is enough to show that $E_> \subset E_2$ since it then follows by taking complements that $E_1 \subseteq E_\bullet$. In Proposition 5.8, we established that any $q \in \mathrm{Im}(B)$ with $\lambda_0(q) > 0$ has two linearly independent, positive solutions of $L_q y = 0$ in $H^1_{\text{loc}}(\mathbb{R})$, so $E_> \subset E_2$.

To prove part (ii), we first note that, by Lemma 5.2, there are compactly supported potentials in E_2, and by the remark above, $\lambda_0(q) = 0$ for such potentials, so $E_2 \cap E_\bullet$ is nonempty. Next, note that $B\left(\mathcal{C}_0^\infty(\mathbb{R})\right) \subset \mathcal{C}_0^\infty(\mathbb{R})$ so $B\left(\mathcal{C}_0^\infty(\mathbb{R})\right) \subset E_\bullet$. On the other hand, by Lemma 5.3, $B\left(\mathcal{C}_0^\infty(\mathbb{R})\right)$ is dense in $\mathrm{Im}(B)$, so $E_\bullet$ is dense in $\mathrm{Im}(B)$. We have already shown that E_1 is dense in $\mathrm{Im}(B)$, so E_1 is also dense in $E_\bullet$ by part (i). The proof of Lemma 5.4 shows that $E_2 \cap E_\bullet$ is dense in $E_\bullet$. □

Remark 5.12. *Note that the map $\Phi : E_\bullet \times \mathbb{R}_{\geq 0} \to \mathrm{Im}(B)$ defined by $(q, c) \mapsto q + c$ is continuous and bijective, but not a homeomorphism. Otherwise, $\Phi(E_\bullet \times \{0\}) \subset \mathrm{Im}(B)$ would be closed, and, as E_1 is dense in $\mathrm{Im}(B)$, we conclude that $E_\bullet = \mathrm{Im}(B)$, a contradiction. The interpretation that E_2 is at least "one dimension larger" than E_1 could therefore be somewhat misleading. Note that the inverse of Φ is given by $\Phi^{-1} : \mathrm{Im}(B) \to E_\bullet \times \mathbb{R}_{\geq 0}$, $q \mapsto (q - \lambda_0(q), \lambda_0(q))$. Hence, Φ^{-1} not being continuous means that $q \mapsto \lambda_0(q)$ is not continuous in $H^{-1}_{\text{loc}}(\mathbb{R})$.*

6. Application to KdV

In this section we apply our results on the Miura map to prove existence of solutions of the Korteweg-de Vries equation in $H^{-1}(\mathbb{R})$ for initial data in the range $\mathrm{Im}(B_0)$ of the Miura map $B_0 : L^2(\mathbb{R}) \to H^{-1}(\mathbb{R})$. We follow the approach of Tsutsumi [46], who proved such an existence result for initial data a positive, finite Radon measure on $\mathbb{R}$. His arguments combined with our results on the Miura map B_0 lead to the following theorem. Recall that, for a real-valued distribution $u \in H^{-1}(\mathbb{R})$, $[u]$ denotes the special integral of u (see Theorem 4.5 and the discussion that precedes it).

Theorem 6.1. *Assume that $u_0 \in \mathrm{Im}(B_0)$. Then there exists a global weak solution of KdV with $u(t) \in \mathrm{Im}\,(B_0)$ for all $t \in \mathbb{R}$. More precisely:*
(i) $u \in L^\infty(\mathbb{R}, H^{-1}(\mathbb{R})) \cap L^2_{\text{loc}}(\mathbb{R}^2)$,
(ii) for all functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, the identity

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(-u\varphi_t - u\varphi_{xxx} + 3u^2 \varphi_x\right)\, dx\, dt = 0$$

holds,
(iii) $\lim_{t \to 0} u(t) = u_0$ in $H^{-1}(\mathbb{R})$, and
(iv) $0 \leq [u(t)] \leq [u_0]$ for all $t \in \mathbb{R}$ and $\lim_{t \to 0} [u(t)] = [u_0]$.

Remark 6.2. *Recall that $u_0 \in \mathrm{Im}(B_0)$ means that $u_0 \in H^{-1}(\mathbb{R})$ with the property that $L_{u_0} \geq 0$ and $[u_0]$ exists. Instead of the assumption for $[u_0]$ to exist, one can equivalently assume that $u_0 = f' + g$ for some functions $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ – see Theorem 1.2 and Proposition 4.3.*

To prove Theorem 6.1, we need to recall a result of Kato [27] and, independently, of Kruzhkov and Faminskiĭ [31] – see also [4] and [18]. Consider the modified Korteweg-de Vries equation (mKdV)

$$\partial_t v = -\partial_x^3 v + 6v^2 \partial_x v \tag{6.1}$$

with initial data

$$v(0) = v_0. \tag{6.2}$$

Theorem 6.3. [27], [31] *Let $v_0 \in L^2(\mathbb{R})$. Then the initial value problem (6.1)-(6.2) has a weak solution in $L^\infty(\mathbb{R}, L^2(\mathbb{R}))$. More precisely, v satisfies:*
(i) $v \in L^\infty(\mathbb{R}, L^2(\mathbb{R})) \cap L^2_{\mathrm{loc}}(\mathbb{R}, H^1_{\mathrm{loc}}(\mathbb{R}))$,
(ii) the identity

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(-v\varphi_t - v\varphi_{xxx} + 2v^3 \varphi_x \right) \, dt \, dx = 0$$

holds for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$,
(iii) $\lim_{t\to 0} v(t) = v_0$ in $L^2(\mathbb{R})$, and
(iv) $\|v(t)\|_{L^2(\mathbb{R})} \le \|v_0\|_{L^2(\mathbb{R})}$ for all $t \in \mathbb{R}$.

The following result improves the one of Tsutsumi [46] by adapting it to our more general setting, and relies on the identity (1.2).

Proposition 6.4. *Let $v = v(t)$ be a solution of (6.1)-(6.2) with $v_0 \in L^2(\mathbb{R})$ and the properties listed in Theorem 6.3. Let $u_0 := v_0' + v_0^2$. Then $u := v' + v^2$ is a solution of KdV. More precisely:*
(i) $u \in L^\infty(\mathbb{R}, H^{-1}(\mathbb{R})) \cap L^2_{\mathrm{loc}}(\mathbb{R}^2)$,
(ii) the identity

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(-u\varphi_t - u\varphi_{xxx} + 3u^2 \varphi_x \right) \, dt \, dx = 0$$

holds for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$,
(iii) $\lim_{t\to 0} u(t) = u_0$ in $H^{-1}(\mathbb{R})$,
(iv) $0 \le [u(t)] \le [u_0]$ for all $t \in \mathbb{R}$, and $\lim_{t\to 0} [u(t)] = [u_0]$.

Proof. Statement (i) follows from Theorem 6.3(i) together with the fact that $B_0 : L^2(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is a bounded, continuous map – see Proposition 2.3. Statement (iii) follows from Theorem 6.3(iii) and the continuity of B_0 whereas the claimed inequality in (iv) follows from Theorem 6.3(iv) and the fact that $[u(t)] = \|v(t)\|^2_{L^2(\mathbb{R})}$ – see formula (4.4). To prove the second statement in (iv), note that by Theorem 6.3(iii), $\lim_{t\to 0} \|v(t)\|_{L^2(\mathbb{R})} = \|v_0\|_{L^2(\mathbb{R})}$. As $[u(t)] = \|v(t)\|^2_{L^2(\mathbb{R})}$, the second statement in (iv) then follows as well. Statement (ii) is proved in [46]. For the convenience of the reader we include a detailed proof of it.

Let $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the smooth mollifier, i.e. a smooth positive function with support in the unit disc in $\mathbb{R}^2$, $\rho(0,0) > 0$, and normalized by $\int_{\mathbb{R}^2} \rho(t,x) \, dt \, dx = 1$. For $\varepsilon > 0$, set

$$\rho_\varepsilon(t,x) := \frac{1}{\varepsilon^2} \rho\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right).$$

Given the solution $v(t)$ of mKdV, define for $(t,x)\in\mathbb{R}^2$

$$\begin{aligned} v_\varepsilon(t,x) &:= (\rho_\varepsilon * v)(t,x) \\ &= \int_{\mathbb{R}^2} \rho\,(t-s,x-y)\;\; v\,(s,y)\;\; ds\;dy. \end{aligned}$$

Note that for any $\varepsilon>0$ and $(t,x)\in\mathbb{R}^2$, $\rho_\varepsilon\,(t-s,x-y)\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ as a function of $(s,y)\in\mathbb{R}^2$. Further define the function $u_\varepsilon\in\mathcal{C}_0^\infty(\mathbb{R}^2)$,

$$u_\varepsilon := \frac{\partial}{\partial x}v_\varepsilon + v_\varepsilon^2.$$

According to (1.2), one has

(6.3)

$$\begin{aligned} \frac{\partial}{\partial t}u_\varepsilon + \frac{\partial^3}{\partial x^3}u_\varepsilon - 6u_\varepsilon\frac{\partial}{\partial x}u_\varepsilon &= \left(\frac{\partial}{\partial x}+2v_\varepsilon\right)\left(\frac{\partial}{\partial t}v_\varepsilon + \frac{\partial^3}{\partial x^3}v_\varepsilon - 6\rho_\varepsilon * \left(v^2\frac{\partial}{\partial x}v\right)\right) \\ &\quad + 6\left(\frac{\partial}{\partial x}+2v_\varepsilon\right)\left(\rho_\varepsilon * \left(v^2\frac{\partial}{\partial x}v\right) - v_\varepsilon^2\frac{\partial}{\partial x}v_\varepsilon\right). \end{aligned}$$

By assumption, v is a weak solution of mKdV, hence

$$\begin{aligned} \frac{\partial}{\partial t}v_\varepsilon + \frac{\partial^3}{\partial x^3}v_\varepsilon - 6\rho_\varepsilon * \left(v^2\frac{\partial}{\partial x}v\right) &= \rho_\varepsilon * \left(\frac{\partial}{\partial t}v + \frac{\partial^3}{\partial x^3}v - 6v^2\frac{\partial}{\partial x}v\right) \\ &= 0. \end{aligned}$$

Multiplying (6.3) by an arbitrary test function $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ and integrating by parts, one obtains

$$\begin{aligned} &\int_{\mathbb{R}^2}\left(-u_\varepsilon\frac{\partial}{\partial t}\varphi - u_\varepsilon\frac{\partial^3}{\partial x^3}\varphi + 3u_\varepsilon^2\frac{\partial}{\partial x}\varphi\right)\;dt\;dx \\ &= 6\int_{\mathbb{R}^2}\left[\rho_\varepsilon * \left(v^2\frac{\partial}{\partial x}v\right) - v_\varepsilon^2\frac{\partial}{\partial x}v_\varepsilon\right]\left(-\frac{\partial}{\partial x}\varphi + 2v_\varepsilon\varphi\right)\;dt\;dx \\ &= 2\int_{\mathbb{R}^2}\left(\rho_\varepsilon * v^3 - v_\varepsilon^3\right)\left(\frac{\partial^2}{\partial x^2}\varphi - 2\varphi\frac{\partial}{\partial x}v_\varepsilon - 2v_\varepsilon\frac{\partial}{\partial x}\varphi\right)\;dt\;dx. \end{aligned} \tag{6.4}$$

By Theorem 6.3(i),

$$\frac{\partial}{\partial x}v_\varepsilon \to \frac{\partial}{\partial x}v \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^2).$$

Lemma 6.5 below together with Theorem 6.3(i) implies that $v\in L^6_{\mathrm{loc}}(\mathbb{R}^2)$. Hence,

$$v_\varepsilon^2\to v^2,\;\; v_\varepsilon^3\to v^3 \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^2).$$

Combining all of this, one obtains

$$\rho_\varepsilon * v^3 - v_\varepsilon^3 = \left(\rho_\varepsilon * v^3 - v^3\right) - \left(v_\varepsilon^3 - v^3\right) \to 0 \;\text{ as }\; \varepsilon\to 0 \;\text{ in }\; L^2_{\mathrm{loc}}(\mathbb{R}^2)$$

and

$$2\varphi\frac{\partial}{\partial x}v_\varepsilon + 2v_\varepsilon\frac{\partial}{\partial x}\varphi \to 2\varphi\frac{\partial}{\partial x}v + 2v\frac{\partial}{\partial x}\varphi \;\text{ in }\; L^2(\mathbb{R}^2).$$

Since $u_\varepsilon = \partial_x v_\varepsilon + v_\varepsilon^2$, it follows that

$$u_\varepsilon \to u \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^2)$$

and, as $\operatorname{supp}\varphi$ is compact,

$$\int_{\mathbb{R}^2}\left(\rho_\varepsilon * v^3 - v_\varepsilon^3\right)\left(\frac{\partial^2}{\partial x^2}\varphi - 2\varphi\frac{\partial}{\partial x}v_\varepsilon - 2v_\varepsilon\frac{\partial}{\partial x}\varphi\right)\;dt\;dx \to 0$$

as $\varepsilon \to 0$. Therefore, taking the limit $\varepsilon \to 0$ in (6.4), we conclude that

$$\int_{\mathbb{R}^2} \left(-u \frac{\partial}{\partial t}\varphi - u \frac{\partial^3}{\partial x^3}\varphi + 3u^2 \frac{\partial}{\partial x}\varphi \right) \, dt \, dx = 0.$$

□

Lemma 6.5. *Let $Q := I \times J$ with $I := [-T, T]$ and $J := [-R, R]$ where $T > 0$ and $R > 0$. Let*

$$\mathcal{B}_{I,J} = L^2\left(I, H^1(J)\right) \cap L^\infty(I, L^2(J))$$

with norm

$$\operatorname{ess\,sup}_{t\in I} \|f(t,\ \cdot\ )\|_{L^2(J)} + \left(\int_I \|f(t,\ \cdot\ )\|^2_{H^1(J)} \, dt \right)^{1/2}.$$

Then, for any $f \in \mathcal{B}_{I,J}$, the inequality

$$\|f\|^6_{L^6(Q)} \le C \operatorname{ess\,sup}_{t\in I} \|f(t,\ \cdot\ )\|^4_{L^2(J)} \cdot \int_I \|f(t,\ \cdot\ )\|^2_{H^1(J)} \, dt \tag{6.5}$$

holds, where $C > 0$ is a constant which depends only on R. In particular, $\mathcal{B}_{I,J}$ embeds continuously into $L^6(Q)$.

Proof. Let us assume first that $f \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. By the standard Sobolev embedding theorem, the space $H^{1/3}(J)$ is densely and continuously embedded in $L^6(J)$ (see e.g. [39], Theorem 1, p 82),

$$\|g\|_{L^6(J)} \le C'_J \|g\|_{H^{1/3}(J)}$$

for all $g \in H^{1/3}(J)$ and some positive constant C'_J. Further, by interpolation, one has for any $g \in H^1(J)$ (see e.g. [39], Remark 2, p 87)

$$\|g\|_{H^{1/3}(J)} \le C''_J \|g\|^{1/3}_{H^1(J)} \|g\|^{2/3}_{L^2(J)}.$$

Setting $C = (C'_J C''_J)^6$, we see that

$$\begin{aligned} \int_{I\times J} |f(t,x)|^6 \, dt \, dx &\le \int_I \left(C'_J \|f(t,\ \cdot\ )\|_{H^{1/3}(J)} \right)^6 \, dt \\ &\le C \int_I \|f\,(t,\ \cdot\ )\|^2_{H^1(J)} \|f\,(t,\ \cdot\ )\|^4_{L^2(J)} \, dt. \end{aligned}$$

By approximation, the above inequality holds for any $f \in \mathcal{B}_{I,J}$. For such f we have

$$\begin{aligned} &\int_I \|f\,(t,\ \cdot\ )\|^2_{H^1(J)} \|f\,(t,\ \cdot\ )\|^4_{L^2(J)} \, dt \\ &\le \operatorname{ess\,sup}_{t\in I} \|f\,(t,\ \cdot\ )\|^4_{L^2(J)} \int_I \|f\,(t,\ \cdot\ )\|^2_{H^1(J)} \, dt. \end{aligned}$$

Combinig the two previous inequalities ends the proof. □

Proof of Theorem 6.1. The proof is the one given in Tsutsumi [46], adapted to our more general setting. By our assumption, $B_0(v_0) = u_0$ for some $v_0 \in L^2(\mathbb{R})$. By Theorem 6.3, there exists a solution $v \in L^\infty\left(\mathbb{R}, L^2(\mathbb{R})\right) \cap L^2_{\mathrm{loc}}\left(\mathbb{R}, H^1_{\mathrm{loc}}(\mathbb{R})\right)$ of (6.1)-(6.2). By Proposition 6.4, $u(t) := B(v(t))$ is a solution of KdV satisfying (i)-(iv). □

Appendix A. Positive Solutions for Square-Well Potentials

In this appendix we present some elementary but important examples of potentials in $\mathrm{Im}(B)$ and the associated positive solutions. Let

$$q_{a,b}(x) = \begin{cases} b^2, & -a < x < a \\ 0, & |x| \geq a \end{cases}$$

where $a, b > 0$. It is easy to see that

$$y_+(x) = \begin{cases} 1/\cosh(ba), & x < -a \\ \cosh(b(x+a))/\cosh(ba), & -a < x < a \\ \left(\cosh(2ab) + b(x-a)\sinh(2ab)\right)/\cosh(ba), & x > a \end{cases} \tag{A.1}$$

and $y_-(x) := y_+(-x)$ are linearly independent positive solutions of $-y'' + q_{a,b}y = 0$.

If $\lambda > 0$ and $b = (\lambda/2a)^{1/2}$ then $\int q_{a,b}(x)\, dx = \lambda$. Taking $a \downarrow 0$ we recover in the limit $q = \lambda\delta$ where δ is the Dirac δ-distribution at $x = 0$. In this limit

$$y_+(x) = \begin{cases} 1 & x < 0 \\ 1 + \lambda x & x > 0 \end{cases}$$

and again $y_-(x) = y_+(-x)$.

Let us determine the preimage of $q = \lambda\delta$ by the Miura map. From the explicit formulas we have $\int_0^\infty y_+(s)^{-2} ds < \infty$ but $\int_{-\infty}^0 y_+(s)^{-2} ds = +\infty$, while the reverse is true for y_-. If H is the Heaviside function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

then the logarithmic derivatives

$$\frac{y_+'(x)}{y_+(x)} = \frac{\lambda H(x)}{1 + \lambda x}, \quad \frac{y_-'(x)}{y_-(x)} = -\frac{y_+'(-x)}{y_+(-x)}$$

belong to $L^2(\mathbb{R})$. Hence

$$B^{-1}(\lambda\delta) = \left\{ (1-\theta)\frac{\lambda H(x)}{1+\lambda x} - \theta\frac{\lambda H(-x)}{1-\lambda x} \mid 0 \leq \theta \leq 1 \right\}.$$

Appendix B. Positive Schrödinger Operators

In this Appendix we provide more information about Schrödinger operators L_q which are positive or, more generally, semibounded below, and have real potentials $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$. Namely, we will show that the corresponding quadratic form (defined on $\mathcal{C}_0^\infty(\mathbb{R})$) is closable and describe the domain of its closure. We will also describe the domain of the corresponding self-adjoint operator. Finally, for strictly positive L_q (such that $\lambda_0(q) > 0$, see (5.8), (5.9)) we construct Green's function and use it to give an alternative proof of Proposition 5.8.

The case of semi-bounded L_q for many purposes is reduced to the case when $L_q \geq 0$ or even to the case when L_q is strictly positive (that is $\lambda_0(q) > 0$) by adding a sufficiently large constant to q. So let us assume first that $L_q \geq 0$. By Theorem 1.1 there exists a function $r \in L^2_{\text{loc}}(\mathbb{R})$ such that $q = B(r)$. Then L_q admits a formal factorization (1.4), i.e., a presentation $L_q = P^+P$, where $P = (\partial_x - r)$ and $P^+ = -(\partial_x + r)$, so that P^+ is the operator formally adjoint to P in $L^2(\mathbb{R})$.

Clearly, P, P^+ are well defined on the space $\mathcal{C}_0^\infty(\mathbb{R})$ which is dense in $L^2(\mathbb{R})$, so that P, P^+ map $\mathcal{C}_0^\infty(\mathbb{R})$ to $L^2(\mathbb{R})$ and

$$(Pu, v) = (u, P^+v), \quad u, v \in \mathcal{C}_0^\infty(\mathbb{R}).$$

It follows that the operators P, P^+ are closable, with the closures which we will denote by $\overline{P}, \overline{P^+}$. They also have adjoint operators in $L^2(\mathbb{R})$, which will be denoted $P^*, (P^+)^*$. These operators are closed extensions of P^+, P respectively. Since $P^*, (P^+)^*$ are closed, we have

$$\overline{P} \subset (P^+)^*, \quad \overline{P^+} \subset P^*. \tag{B.1}$$

Lemma B.1. *(i) We have*

$$\overline{P} = (P^+)^*, \quad \overline{P^+} = P^*. \tag{B.2}$$

(ii) The domains of the operators in (B.2) *are as follows:*

$$\mathfrak{D}(\overline{P}) = \left\{ u \in L^2(\mathbb{R}) \cap W^{1,1}_{\text{loc}}(\mathbb{R}) : Pu \in L^2(\mathbb{R}) \right\}, \tag{B.3}$$

$$\mathfrak{D}(\overline{P^+}) = \left\{ v \in L^2(\mathbb{R}) \cap W^{1,1}_{\text{loc}}(\mathbb{R}) : P^+v \in L^2(\mathbb{R}) \right\}, \tag{B.4}$$

where the operators P, P^+ are applied in the usual distributional sense.

(iii) $\mathcal{C}_0^\infty(\mathbb{R})$ is an operator core for each of the operators in (B.2).

Proof. It is easy to see that the right-hand sides in (B.3), (B.4) coincide with the domains of the adjoint operators $(P^+)^*, P^*$ respectively. Indeed, the relation $(P^+)^*u = f$ means that $u, f \in L^2(\mathbb{R})$ and for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$

$$(u, P^+\varphi) = -(u, \partial_x\varphi) - (u, r\varphi) = (f, \varphi).$$

Since $ru \in L^1_{\text{loc}}(\mathbb{R})$, this is equivalent to $\partial_x u - ru = f$, where ∂_x is applied in the sense of distributions. It follows that $\partial_x u = f + ru \in L^1_{\text{loc}}(\mathbb{R})$, hence $u \in W^{1,1}_{\text{loc}}(\mathbb{R})$ or, equivalently, u is absolutely continuous. This means that the right hand side of (B.3) coincides with $\mathfrak{D}((P^+)^*)$. The same argument applies to the operator P^* and the right-hand side of (B.4).

Taking into account the inclusions (B.1), we see that to establish all statements of the lemma, it suffices to show that the right hand sides of (B.3), (B.4) belong to the domains of $\overline{P}, \overline{P^+}$ respectively. This is easily done by use of Friedrichs' mollifiers. It is essentially a special case of Friedrichs' [13] well-known result on equality of weak and strong extensions of differential operators, but we give the proof for the reader's convenience. We will give the arguments for P (the arguments for P^+ are the same).

So let us assume that

$$u \in L^2(\mathbb{R}) \cap W^{1,1}_{\text{loc}}(\mathbb{R}), Pu \in L^2(\mathbb{R}). \tag{B.5}$$

We need to show that u may be approximated by a sequence $\{u_n\}_{n\geq1}$ from $\mathcal{C}_0^\infty(\mathbb{R})$ with $u_n \to u$ and $Pu_n \to Pu$ in $L^2(\mathbb{R})$. First, we show that it suffices to consider u satisfying (B.5) and additionally having compact support. To this end, take $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$. Then χu also satisfies (B.5) and $P(\chi u) = \chi' u + \chi Pu$. If $\chi_n \in \mathcal{C}_0^\infty(\mathbb{R})$

satisfies the conditions $0 \le \chi_n \le 1$, $\chi_n(x) = 1$ for $|x| \le n$, $\chi_n(x) = 0$ for $|x| \ge n+1$, and $|\chi_n'(x)| \le 2$, then $\chi_n u \to u$ in $L^2(\mathbb{R})$ as $n \to \infty$. Moreover, $\chi_n' u \to 0$ and $\chi_n Pu \to Pu$ in $L^2(\mathbb{R})$ as $n \to \infty$, so $P(\chi_n u) = \chi_n' u + \chi_n Pu \to Pu$ in $L^2(\mathbb{R})$ as $n \to \infty$ Thus, we may assume that u has compact support.

Given u satisfying (B.5) and having compact support, we now use Friedrichs mollifiers to construct a sequence of approximants from $\mathcal{C}_0^\infty(\mathbb{R})$. Let $j \in \mathcal{C}_0^\infty(\mathbb{R})$ be a nonnegative function with $\int j(x)\, dx = 1$, and, for any $k \in \mathbb{N}$, let $j_k(x) = kj(kx)$, and let $u_k = u * j_k$. Clearly $u_k \in \mathcal{C}_0^\infty(\mathbb{R})$ and $u_k \to u$ in $L^2(\mathbb{R})$. We claim that $Pu_k \to Pu$ in $L^2(\mathbb{R})$. Since $u \in W^{1,1}_{\text{comp}}(\mathbb{R})$ and $r \in L^2_{\text{loc}}(\mathbb{R})$, it follows that $ru \in L^1(\mathbb{R})$. Moreover, as u satisfies (B.5) and the support of u is compact, $Pu \in L^1(\mathbb{R})$. Therefore $u' = Pu + ru$ belongs to $L^1(\mathbb{R})$. Hence, u is a bounded, continuous function which shows that $ru \in L^2(\mathbb{R})$. As a consequence, $u' = Pu + ru \in L^2(\mathbb{R})$. Thus $u_k' \to u'$ in $L^2(\mathbb{R})$, $u_k \to u$ in $L^\infty(\mathbb{R})$ as $k \to \infty$ and so

$$Pu_k - Pu = (u_k' - u') + r\,(u_k - u)$$

converges to zero in $L^2(\mathbb{R})$ as $k \to \infty$. □

Now let us recall a classical theorem of von Neumann [47] (see also [38], Theorem XI.23) which asserts that if A is a closed densely defined operator in a Hilbert space, then the (generally unbounded) operator $H = A^*A$ is self-adjoint. Here the domain of A^*A is naturally defined as

$$\mathfrak{D}(A^*A) = \{u \in \mathfrak{D}(A), Au \in \mathfrak{D}(A^*)\}.$$

(Note that an essentially inverse statement also holds: if two densely defined operators A, A^+ are formally adjoint, that is

$$(Au, v) = (u, A^+ v), \quad u \in \mathfrak{D}(A), v \in \mathfrak{D}(A^+),$$

and A^+A is essentially self-adjoint, then the closures $\overline{A}, \overline{A^+}$ are adjoint to each other; see the appendix to [42].)

The following lemma is well-known.

Lemma B.2. *Let A be a closed densely defined operator in a Hilbert space, and $H = A^*A$. Denote by $\mathfrak{t}_H$ the quadratic form of H, and let $\mathfrak{D}(\mathfrak{t}_H)$ be its domain i.e. $\mathfrak{D}(\mathfrak{t}_H) = \mathfrak{D}(H^{1/2})$. Then $\mathfrak{D}(\mathfrak{t}_H) = \mathfrak{D}(A)$ and*

$$\mathfrak{t}_H(u, u) = \|Au\|^2, \quad u \in \mathfrak{D}(A).$$

Proof. Take the polar decomposition $A = U|A|$, where $|A| = (A^*A)^{1/2} = H^{1/2}$, and U partial isometry with $\operatorname{Ker} U = \operatorname{Ker} A$ (see e.g. Sect. VIII.9 in [38]). It remains to notice that $\mathfrak{D}(A) = \mathfrak{D}(|A|)$ (because U is bounded), and

$$\mathfrak{t}_H(u, u) = \|H^{1/2}u\|^2 = \||A|u\|^2 = \|Au\|^2, \quad u \in \mathfrak{D}(A),$$

because U is an isometry on the range of $|A|$. □

Note that a positive self-adjoint operator H is uniquely defined by its (positive, closed) quadratic form (see e.g. Theorem VIII.15 in [38]).

Taking the quadratic form $\mathfrak{t}_q$, corresponding to a potential $q \in H^{-1}_{\text{loc}}(\mathbb{R})$ and assuming that it is positive (or, more generally, semi-bounded below), we can construct a unique self-adjoint operator H with this form. It follows from the considerations above that when the form $\mathfrak{t}_q$ is positive, we can write this operator in the form $H = P^*\overline{P}$, and the domain of H is

$$\mathfrak{D}(H) = \{u \in L^2(\mathbb{R}),\ (\partial_x - r)u \in L^2(\mathbb{R}),\ (\partial_x + r)\,[(\partial_x - r)u] \in L^2(\mathbb{R})\}, \tag{B.6}$$

where $r \in L^2_{\rm loc}(\mathbb{R})$ is a solution of the Riccati equation $r' + r^2 = q$.

In case when L_q is semibounded below but not positive, the arguments given above should be applied to the operator $L_q + c$ with $c > 0$ such that $L_q + c \geq 0$, with subsequent subtracting of the same constant c from the resulting operator (which does not change the domain of the operator). The resulting operator H will not depend of the choice of c because different choices of c lead to the same (closed) quadratic form of the resulting operator.

The following lemma simplifies calculation of Hu if we know that $u \in \mathfrak{D}(H)$.

Lemma B.3. *Let $q \in H^{-1}_{\rm loc}(\mathbb{R})$ be such that $L_q \geq 0$. Then H can be extended to a linear operator $\widetilde{L}_q$ with the domain*

$$\mathfrak{D}(\widetilde{L}_q) = \{u \in H^1_{\rm loc}(\mathbb{R}) \cap L^2(\mathbb{R}),\ \widetilde{L}_q u \in L^2(\mathbb{R})\}, \tag{B.7}$$

where $\widetilde{L}_q$ is $-\partial_x^2 + q$ applied in the sense of distributions, i.e. both ∂_x^2 and q act as linear continuous operators $H^1_{\rm loc}(\mathbb{R}) \to H^{-1}_{\rm loc}(\mathbb{R})$ (∂_x^2 acts as the distributional derivative, and q acts as a multiplier in these spaces).

Proof. If $u \in \mathfrak{D}(H)$ (as described by (B.6)) then $u \in L^2(\mathbb{R})$ and $f := u' - ru \in L^2(\mathbb{R})$. Since $ru \in L^1_{\rm loc}(\mathbb{R})$, we see that $u' \in L^1_{\rm loc}(\mathbb{R})$, hence $u \in W^{1,1}_{\rm loc}(\mathbb{R})$, i.e. u is absolutely continuous. But then $ru \in L^2_{\rm loc}(\mathbb{R})$, and $u \in H^1_{\rm loc}(\mathbb{R})$. Now we can conclude that

$$-(\partial_x + r)(\partial_x - r)u = (-\partial_x^2 + q)u,$$

where all operations should be applied in the distributional sense. We proved that $\mathfrak{D}(H) \subset \mathfrak{D}(\widetilde{L}_q)$ and H, applied as a factorized operator (see (1.4)), is a restriction of $\widetilde{L}_q$. $\square$

Remark B.4. *In particular, we can apply H on $\mathfrak{D}(H)$ as $-\partial_x^2 + q$ applied termwise, which is usually easier than to apply it in the factorized form. To illustrate it, note that it may easily happen that $\mathfrak{D}(\widetilde{L}_q)$ does not contain any function $u \in \mathcal{C}_0^\infty(\mathbb{R})$ (except $u \equiv 0$). This is true e.g. for the potential*

$$q(x) = \sum_{k=1}^\infty c_k \delta(x - x_k), \quad \sum_{k=1}^\infty c_k < \infty,$$

where $c_k > 0$ for all k, and the set $\{x_k\}_{k=1}^\infty$ is dense in $\mathbb{R}$. Therefore, the same is true for $\mathfrak{D}(H)$.

Remark B.5. *The operator $\widetilde{L}_q$ may be defined on the domain* (B.7) *even without semi-boundednes requirement. But in this case the resulting operator (called usually "maximal operator") in $L^2(\mathbb{R})$ will not necessarily be self-adjoint even if $q \in \mathcal{C}^\infty(\mathbb{R})$ (see e.g. Sect. X.1 in* [38] *or Sect. II.1 and II.4.2 in* [3]*).*

Now, note that $\lambda_0(q)$ (as defined in (5.8)) is the bottom of the spectrum of the self-adjoint operator H. So, if $\lambda_0(q) > 0$, then there exists a bounded, everywhere defined linear operator $T := H^{-1}$ in $L^2(\mathbb{R})$. It maps $L^2(\mathbb{R})$ onto $\mathfrak{D}(H)$. We will now analyze the properties of the Schwartz kernel of T, which is Green's function for the operator H, where H is an arbitrary, but fixed Schrödinger operator with a real potential $q \in H^1_{loc}(\mathbb{R}^n)$ such that $\lambda_0(q) > 0$.

Lemma B.6. *The following estimates hold for any bounded open interval $I \subset \mathbb{R}$ and any $u \in H^1_{\mathrm{loc}}(I)$:*

$$\|u\|_{L^\infty(I)} \le |I|^{-1/2} \|u\|_{L^2(I)} + \|u'\|_{L^1(I)}\,, \tag{B.8}$$

and

$$\|u'\|_{L^1(I)} \le |I|^{1/2} \|Pu\|_{L^2(I)} + \|r\|_{L^2(I)} \|u\|_{L^2(I)}\,. \tag{B.9}$$

where $|I|$ means the length of the interval I.

Proof. Both estimates are well-known but we provide the proofs for the convenience of the reader. We start with

$$u(x) - u(y) = \int_y^x u'(s)ds, \quad x, y \in I,$$

which implies

$$|u(x) - u(y)| \le \|u'\|_{L^1(I)},$$

hence

$$|u(x)| \le |u(y)| + \|u'\|_{L^1(I)},$$

and

$$\|u\|_{L^\infty(I)} \le |u(y)| + \|u'\|_{L^1(I)},$$

for all $y \in I$. Integrating with respect to $y \in I$ and dividing by I, we get

$$\|u\|_{L^\infty(I)} \le |I|^{-1}\|u\|_{L^1(I)} + \|u'\|_{L^1(I)}.$$

Applying the Cauchy-Schwarz inequality in the first term in the right hand side, we obtain (B.8).

From $u' = Pu + ru$, taking L^1-norms of both sides and using the Cauchy-Schwarz inequality we obtain

$$\|u'\|_{L^1(I)} \le \|Pu\|_{L^1(I)} + \|ru\|_{L^1(I)} \le |I|^{1/2}\|Pu\|_{L^2(I)} + \|r\|_{L^2(I)}\|u\|_{L^2(I)},$$

which proves (B.9). □

Lemma B.7. *Let us assume that $q \in H^{-1}_{loc}(\mathbb{R})$ with $\lambda_0(q) > 0$. Then*
1) The operator $\overline{P}T$ is defined everywhere in $L^2(\mathbb{R})$ and $\|\overline{P}T\| = \lambda_0(q)^{-1/2}$.
2) $\overline{P}TP^* = I$ on $\mathfrak{D}(P^*)$.

Proof. 1) Since $P^*(\overline{P}T) = (P^*\overline{P})T = I$, the operator $\overline{P}T$ is everywhere defined in $L^2(\mathbb{R})$. Also, for any $u \in L^2(\mathbb{R})$,

$$\left\|\overline{P}Tu\right\|^2 = (\overline{P}Tu, \overline{P}Tu) = (TP^*\overline{P}Tu, u) = (Tu, u) = \|T^{1/2}u\|^2,$$

so the first statement immediately follows. (In fact, the presentation $\overline{P}T = UT^{1/2}$, with $U = \overline{P}T^{1/2}$, is the polar decomposition of $\overline{P}T$.)

2) For any $u, v \in \mathfrak{D}(P^*\overline{P})$ we have

$$((\overline{P}TP^*)\overline{P}u, \overline{P}v) = (\overline{P}T(P^*\overline{P})u, \overline{P}v) = (\overline{P}u, \overline{P}v),$$

so $\overline{P}TP^*u = u$ for all $u \in \mathfrak{D}(P^*\overline{P})$. It remains to recall that $\mathfrak{D}(P^*\overline{P})$ is dense in $L^2(\mathbb{R})$. □

Heuristically, Green's function $G = G(x,y)$ should be given by

$$G(x,y) = (T\delta(\cdot - y))(x). \tag{B.10}$$

So it is expected to satisfy

$$(-\partial_x^2 + q(x))G(x,y) = \delta(x-y), \tag{B.11}$$

and be the Schwartz kernel of a bounded linear operator in $L^2(\mathbb{R})$. More precisely, we will prove:

Lemma B.8. *Let us assume that $q \in H^{-1}_{loc}(\mathbb{R})$ with $\lambda_0(q) > 0$. Then there exists a measurable, real-valued function $G(x,y)$ on $\mathbb{R}\times\mathbb{R}$ with the following properties:*
(i) For any bounded interval I in $\mathbb{R}$, there is a positive constant $C(I)$ so that

$$\sup_{x\in I}\left(\int G(x,y)^2\ dy\right)^{1/2} \le C(I)$$

and the map $x \mapsto G(x,\ \cdot\ )$ is Hölder continuous of order $1/2$ as a mapping from I into $L^2(\mathbb{R})$.
(ii) For any $f \in L^2(\mathbb{R})$,

$$(Tf)(x) = \int G(x,y)\ f(y)\ dy$$

(iii) $G(x,y) = G(y,x)$ almost everywhere.
(iv) For each fixed x, the function $G(x,\ \cdot\ )$ belongs to $\mathfrak{D}(\overline{P})$.
(v) For any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and any $x \in \mathbb{R}$, $\left(\overline{P}G(x,\cdot), \overline{P}\varphi\right) = \varphi(x)$.

Remark B.9. *Part (v) states that $G(x,y)$, viewed as a function of y with x as parameter, solves the equation $L_q G(x,y) = \delta_x(y)$ in distribution sense.*

Proof of Lemma B.8. In what follows, $r \in L^2_{loc}(\mathbb{R})$ is fixed and satisfies $q = r' + r^2$, I denotes a bounded interval in $\mathbb{R}$, and $C(I)$ denotes a generic constant depending on $|I|$ and r. Its value may vary from line to line.

We will make repeated use of the following observation, based on Lemma B.6. If $\psi \in \mathcal{D}(\overline{P})$ and I is a bounded interval, then by Lemma B.6,

$$\sup_{x\in I}|\psi(x)| \le C(I)\left(\|\psi\|_{L^2(I)} + \left\|\overline{P}\psi\right\|_{L^2(\mathbb{R})}\right). \tag{B.12}$$

Using the boundedness of ψ and the fact that $\overline{P}\psi \in L^2(\mathbb{R})$, we can then deduce that

$$\begin{aligned}\|\psi'\|_{L^2(I)} &\le \left\|\overline{P}\psi\right\|_{L^2(\mathbb{R})} + \|r\|_{L^2(I)}\|\psi\|_{L^\infty(I)} \\ &\le C(I)\left(\|\psi\|_{L^2(\mathbb{R})} + \left\|\overline{P}\psi\right\|_{L^2(\mathbb{R})}\right).\end{aligned} \tag{B.13}$$

Since, by Lemma B.7, $\left\|\overline{P}T\psi\right\|_{L^2(\mathbb{R})} \le C\|\psi\|_{L^2(\mathbb{R})}$, it follows from (B.12) and (B.13) that for any $\psi \in L^2(\mathbb{R})$, one has $T\psi \in H^1_{\text{loc}}(\mathbb{R})$ with

$$\sup_{x\in I}|(T\psi)(x)| \le C(I)\|\psi\|_{L^2(\mathbb{R})} \tag{B.14}$$

and

$$\int_I \left|\frac{d}{dx}(T\psi)(x)\right|^2\ dx \le C(I)\|\psi\|^2_{L^2(\mathbb{R})}. \tag{B.15}$$

In particular, for each x, the map $x \mapsto (T\psi)(x)$ is a bounded linear functional on $L^2(\mathbb{R})$. It follows from the Riesz representation theorem that there is an element G_x of $L^2(\mathbb{R})$ with

$$(T\psi)(x) = (\psi, G_x).$$

We claim that, also, the map $I \ni x \mapsto G_x \in L^2(\mathbb{R})$ is Hölder continuous of order 1/2. To see this, we use (B.15) together with the Cauchy-Schwarz inequality to conclude that for x and y belonging to I,

$$|(T\psi)(x) - (T\psi)(y)| \leq C(I)\,|x-y|^{1/2}\,\|\psi\|_{L^2(\mathbb{R})}$$

and thus

$$\sup\left\{|(\psi, G_x - G_y)| : \psi \in L^2(\mathbb{R}),\ \|\psi\|_{L^2(\mathbb{R})} = 1\right\} \leq C(I)\,|x-y|^{1/2}.$$

This proves the required Hölder continuity. It follows that the map $x \mapsto G_x$ is a weakly measurable map from I into $L^2(\mathbb{R})$ with $\|G_x\|_{L^2(\mathbb{R})}$ bounded uniformly in $x \in I$, so that $x \mapsto G_x$ may be regarded as an element of the space $L^2(I; L^2(\mathbb{R}))$ consisting of weakly measurable, square-integrable functions on I taking values in $L^2(\mathbb{R})$. By Theorem III.11.17 of [11], there is a measurable function $G_I(x,y)$ on $I \times \mathbb{R}$ with the property that $G_I(x, \,\cdot\,) = G_x$ for every $x \in I$. As

$$(\varphi, T\psi) = \int_{I\times\mathbb{R}} \varphi(x) G_I(x,y) \psi(y)\ dy\ dx$$

for any $\varphi \in L^\infty(I)$ and $\psi \in L^2(\mathbb{R})$, it is easy to see that for any bounded intervals I and J with $I \subset J$, the restriction of G_J to $I \times \mathbb{R}$ equals G_I almost everywhere with respect to product measure on $I \times \mathbb{R}$. Taking a sequence of bounded intervals $\{I_n\}$ with $I_n \nearrow \mathbb{R}$ as $n \to \infty$, we can construct a measurable function G on $\mathbb{R} \times \mathbb{R}$ that obeys properties (i) and (ii). Property (iii) follows from the symmetry of T.

To prove property (iv), let $\varphi \in \mathcal{D}(P^*)$ and note that

$$(G_x, P^*\varphi) = (TP^*\varphi)(x).$$

By Lemma B.7, $\left\|\overline{P}TP^*\varphi\right\|_{L^2(\mathbb{R})} \leq \|\varphi\|_{L^2(\mathbb{R})}$ holds. Hence $TP^*\varphi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and, for any bounded interval I,

$$\sup_{x\in I} |(TP^*)(x)| \leq C(I)\,\|\varphi\|_{L^2(\mathbb{R})}$$

so that

$$|(G_x, P^*\varphi)| \leq C(I)\,\|\varphi\|_{L^2(\mathbb{R})}$$

for any $\varphi \in \mathcal{D}(P^*)$. This shows that $G_x \in \mathcal{D}(P^{**}) = \mathcal{D}(\overline{P})$, proving (iv).

Finally, to prove (v), let $\varphi \in \mathcal{D}(H)$ and compute

$$\begin{aligned}
\varphi(x) &= (TH\varphi)(x) \\
&= (H\varphi, G_x) \\
&= (\overline{P}\varphi, \overline{P}G_x).
\end{aligned}$$

Since $\mathcal{D}(H)$ is dense in $\mathcal{D}(\overline{P})$ and point evaluations are continuous in $\mathcal{D}(\overline{P})$, it follows that $\varphi(x) = (\overline{P}\varphi, \overline{P}G_x)$ for all $\varphi \in \mathcal{D}(\overline{P})$. Since $\mathcal{C}_0^\infty(\mathbb{R}) \subset \mathcal{D}(\overline{P})$, (v) is proved. $\square$

To construct positive solutions from Green's function, we will need the following lemma.

Lemma B.10. *Suppose $\lambda_0(q) > 0$ and that $y \in H^1_{\mathrm{loc}}(\mathbb{R})$, $L_q y = 0$, and either*
(i) $y \in L^2(0,\infty)$ and $Py \in L^2(0,\infty)$, or
(ii) $y \in L^2(-\infty,0)$ and $Py \in L^2(-\infty,0)$.
Then, either y has no zeros on $\mathbb{R}$ or y is identically zero on $\mathbb{R}$.

Proof. We will give the proof assuming (i) holds since the proof assuming (ii) holds is similar. Suppose that $y \in H^1_{\mathrm{loc}}(\mathbb{R})$ solves $L_q y = 0$, (i) holds, and $y(x_0) = 0$ for some $x_0 \in \mathbb{R}$. We will assume without loss that $x_0 = 0$. By assumption (i), the function

$$w(x) = \begin{cases} y(x), & 0 \le x < \infty \\ 0, & x < 0 \end{cases}$$

belongs to $L^2(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\mathbb{R})$, and $Pw \in L^2(\mathbb{R})$, hence $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$. It follows from Lemma B.1 that $w \in \mathfrak{D}(\overline{P})$.

We claim that there is a sequence $\{\varphi_n\}$ from $\mathcal{C}_0^\infty(\mathbb{R})$ with support contained in $(0,\infty)$ so that $\varphi_n \to w$ and $P\varphi_n \to \overline{P}w$ in $L^2(\mathbb{R})$. If so then, on the one hand,

$$\left(P\varphi_n, \overline{P}w\right) = 0 \tag{B.16}$$

since $L_q y = 0$ and $L_q y = L_q w$ as distributions on $(0,\infty)$. Taking limits in (B.16) as $n \to \infty$ we have

$$(\overline{P}w, \overline{P}w) = 0, \tag{B.17}$$

hence $\overline{P}w = 0$. On the other hand, we have

$$(P\varphi_n, P\varphi_n) \ge \lambda_0(q)\, \|\varphi_n\|^2 \tag{B.18}$$

where $\lambda_0(q) > 0$. Taking limits as $n \to \infty$ in (B.18) and using (B.17), we conclude that $w = 0$. It follows from the uniqueness of solutions to $L_q y = 0$ with prescribed initial data that $y = 0$ identically.

Thus, it remains to prove the existence of a sequence $\{\varphi_n\}$ with the claimed properties. First, we show that w may be approximated by functions which vanish identically near $x = 0$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $0 \le \chi(x) \le 1$, $\chi(x) = 0$ for $x \le 1$, $\chi(x) = 1$ for $x \ge 2$, and $|\chi'(x)| \le 2$ for all $x \in \mathbb{R}$. Let $\chi_\varepsilon(x) = \chi(x/\varepsilon)$. The functions $w_\varepsilon(x) = \chi_\varepsilon(x) w(x)$ converge to w in $L^2(\mathbb{R})$ as $\varepsilon \to 0$ by dominated convergence. We claim that, also, $Pw_\varepsilon \to \overline{P}w$ in $L^2(\mathbb{R})$. To see this, compute

$$Pw_\varepsilon = \chi_\varepsilon'(x) w(x) + \chi_\varepsilon(x)(\overline{P}w)(x). \tag{B.19}$$

The second term in the right-hand side of (B.19) converges to $\overline{P}w$ in $L^2(\mathbb{R})$ by dominated convergence while the first one converges to 0 in $L^2(\mathbb{R})$ by the following reasons. Observing that

$$\int \chi_\varepsilon'(x)^2\, |w(x)|^2\, dx \le \frac{4}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} \left(\int_0^x |w'(t)|\, dt \right)^2 dx \le 4 \int_0^{2\varepsilon} |w'(t)|^2\, dt,$$

we conclude

$$\int \chi_\varepsilon'(x)^2\, |w(x)|^2\, dx \to 0 \ \text{ as } \varepsilon \to 0.$$

Thus $Pw_\varepsilon \to \overline{P}w$ in $L^2(\mathbb{R})$.

Letting $\varepsilon = 1/n$, the function $w_{1/n}$ has support in $[1/n, \infty)$. We can use smooth cut-off functions and Friedrichs mollifiers as in the proof of Lemma B.1 to find a $\mathcal{C}_0^\infty$ function φ_n with support in $[1/(2n), \infty)$ so that $\left\|\varphi_n - w_{1/n}\right\| < 1/n$ and $\left\|P\varphi_n - \overline{P}w_{1/n}\right\| < 1/n$. In this way we obtain a sequence $\{\varphi_n\}$ from $\mathcal{C}_0^\infty(0,\infty)$ so that $\|\varphi_n - w\| \to 0$ and $\left\|P\varphi_n - \overline{P}w\right\| \to 0$ as $n \to \infty$. $\square$

As an application of the results obtained in this appendix, we give an alternative proof of Proposition 5.8

Proof of Proposition 5.8. We claim that there exists an $x \in \mathbb{R}$ so that $y \mapsto G(x,y)$ does not vanish identically on (x,∞). If not then $G(x,y)=0$ for all (x,y) with $y>x$ and hence, by Lemma B.8(iii), for all $y \neq x$. Therefore $G(x,y)=0$ a.e., a contradiction. Now choose such an x. Then the function $\psi_+(y)=G(x,y)$ for $y>x$ is not identically zero on (x,∞). From Lemma B.8(i), (iv), and (v), $\psi_+(y) \in L^2(x,\infty)$, $P\psi_+ \in L^2(x,\infty)$, and $L_q\psi_+ = 0$ for $y>x$. Let Q be an antiderivative of q and let $\{y_+, u_+\}$ be the unique solution to the system (2.7) with initial data $y_+(x+1)=\psi_+(x+1)$ and $(u_+)(x+1)=(\psi_+ - Q\psi_+)(x+1)$. Then y_+ coincides with ψ_+ on (x,∞), so y_+ and Py_+ belong to $L^2(0,\infty)$. It follows from Lemma B.10 that y_+ has no zeros, so by changing signs if necessary we conclude that $y_+ \in L^2(0,\infty)$ and y_+ is strictly positive on $\mathbb{R}$. A similar construction considering the function $\psi_-(y)=G(x,y)$ for some $x \in \mathbb{R}$ and $y<x$ leads to a strictly positive solution y_- of $L_q y = 0$ with $y_- \in L^2(-\infty,0)$. If y_+ and y_- were linearly dependent, then after multiplying one of them by an appropriate constant, we would obtain a function ψ in the domain of H, ψ not identically zero, with $H\psi=0$, which is impossible since $\lambda_0(q)>0$. Thus y_+ and y_- are linearly independent. □

Appendix C. Related Work

The Miura map was introduced by Miura [34], [35] and played an important role in the search of integrals of motion for the Korteweg-de Vries equation. Miura discovered that his map takes smooth solutions of mKdV to smooth solutions of KdV. Hence it can serve as a tool to derive results on the initial value problem for KdV from results on the initial value problem for mKdV – see e.g. [46]. Despite the fact that the Miura map is not one-to-one, when considered, for example, as a map between appropriate Sobolev spaces, it is also possible to use it to derive results for the initial value problem of mKdV from results of the initial value problem of KdV – see e.g. [16], [9], [26].

Miura map on the circle: Motivated by earlier work of Ambrosetti and Prodi [2] on certain nonlinear elliptic boundary value problems, McKean and Scovel [33] studied - among other nonlinear maps - the Miura map on the circle $\mathbb{T}$. They exhibited a global fold structure for the Miura map when viewed as a map from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$. For further results in this direction, see Bueno and Tomei [7]. Later, Korotyaev [29], [30] and Kappeler and Topalov [24] extended the global fold picture to the Miura map from periodic functions in $L^2(\mathbb{T})$ to $H^{-1}(\mathbb{T})$. Kappeler and Topalov proved existence and well-posedness of solutions to the mKdV equation with initial data in $L^2(\mathbb{T})$ [26], using [24] and their results on the initial value problem for the periodic KdV equation established in [25].

Miura map on the line: On the line, the Miura map and related topics have also been investigated extensively, and not exclusively with a view towards applications for solving the initial value problem of KdV or mKdV.

• Positive solutions of Schrödinger equations or more generally of second order elliptic equations – in particular in connection with spectral properties of the corresponding operators – have been extensively studied in various settings. We only mention the result, referred to as Allegretto-Piepenbrink theorem in [10], Theorem 2.12 or in [44], section C.8. This theorem states that for potentials $q \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, satisfying some additional conditions, $(-\Delta + q)u = \lambda u$ has a nonzero solution u

(in the sense that $u \in W^{2,1}_{\text{loc}}(\mathbb{R}^n)$ and $qu \in L^1_{\text{loc}}(\mathbb{R}^n)$), which is nonnegative everywhere, if and only if $\inf(\text{spec}(-\Delta + q)) \geq \lambda$. See [10] or [44] for further details and references to the papers of Allegretto and of Piepenbrink as well as additional references. In the one-dimensional case at hand, the equivalence of the statements (ii) and (iii) of Theorem 1.1 for potentials $q \in L^1_{\text{loc}}(\mathbb{R})$ is well known – see [19], Theorems XI.6.1 and XI.6.2 and Corollary XI.6.1, [36], Appendix 1, or [17], Theorem 3.1. Thus Theorem 1.1 as stated above shows in particular that this equivalence continues to hold for $q \in H^{-1}_{\text{loc}}(\mathbb{R})$.

• With regard to the characterization of the image of the Miura map B_0 , we mention the result of Ablowitz et. al. [1] which characterizes the image of Schwartz space by B_0 in terms of the scattering data of these potentials as well as the result of Tsutsumi [46], stating that any finite, positive Radon measure is in the image of the Miura map $B_0 : L^2(\mathbb{R}) \to H^{-1}(\mathbb{R})$. Further, the case where q is continuous is treated by Hartman [19], Chapter XI.7, Lemma 7.1. The result stated in Theorem 1.2 sharpens all these results and puts them into a broader perspective.

• The dichotomy described in Theorem 1.3 has another interpretation which does not involve the Miura map at all: Murata [36], Appendix 1, describes the dichotomy stated in Theorem 1.3 – again for potentials in $q \in L^1_{\text{loc}}(\mathbb{R})$ – in terms of the notion of subcritical, critical, and supercritical potentials, where in his terminology (i) q is called subcritical if L_q has a positive Green's function, (ii) q is called critical if $L_q \geq 0$ and does *not* have a positive Green's function , and (iii) q is called supercritical if L_q is *not* nonnegative. See Simon [43] for an alternative notion of subcritical and critical potentials. In [36], Theorem A.5, Murata shows that (i) $q \in L^1_{\text{loc}}(\mathbb{R})$ is subcritical iff $L_q y = 0$ admits two linearly independent positive solutions in $W^{2,1}_{\text{loc}}(\mathbb{R})$ and that (ii) $q \in L^1_{\text{loc}}(\mathbb{R})$ is critical iff $L_q y = 0$ has up to scaling one positive solution $y \in W^{2,1}_{\text{loc}}(\mathbb{R})$. These results of Murata for one-dimensional Schrödinger operators were later extended by Gesztesy and Zhao [17], Theorem 3.6, to more general Sturm-Liouville operators. Our results on the dichotomy for Schrödinger operators obtained in this paper extend the results of Murata (and of Gesztesy and Zhao) in two directions. First, we consider potentials which are real-valued distributions in a Sobolev space with negative index of smoothness $\beta \geq -1$. Hence they are not necessarily functions. Second, we describe geometric aspects of the dichotomy: see Theorem 1.3 and Theorem 5.6.

Schrödinger operators with singular potentials: Recently, the operators L_q, considered on an interval (a, b), with potential q in a Sobolev space with negative index of smoothness, have been studied by various authors. In particular, we mention the paper [41] where different approaches to define the operator L_q are discussed in detail and asymptotics for the eigenvalues and eigenfunctions of these operators are obtained. See also [20], [21], [23], [29], [30], [37], [40], [41] as well as [41] for further references.

Initial value problem for KdV: The initial value problem for KdV on the line has been extensively studied. We only mention that, based on the works of Bourgain [5], [6], it has been proved by Kenig-Ponce-Vega [28] that KdV is locally uniformly C^0 well-posed on $H^s(\mathbb{R})$ for $s > -3/4$ and later, by Colliander, Keel, Staffilani, Takaoka and Tao [9], that KdV is *globally* uniformly C^0 well-posed on $H^s(\mathbb{R})$ for $s > -3/4$. An existence result for the limiting case $s = -3/4$ has been obtained by Christ, Colliander and Tao [8]. Beside the work of Tsutsumi already mentioned above on solutions of KdV with positive Radon measures as initial data, it has been shown in [22] that for measures of bounded variation with sufficient decay at infinity as initial data, there exists a *classical* solution for $t > 0$.

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