SEMICLASSICAL ASYMPTOTICS ON COVERING MANIFOLDS AND MORSE INEQUALITIES.

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Introduction.

1. This paper contains a review of the results which can be obtained by applying the Witten deformation method [W1] and general semi-classical asymptotics to the case of regular covering manifolds. Many ideas and some of the results described here were known before (e.g. the Morse-type inequalities with the von Neumann Betti numbers which were proved in [N-S1]). However the proofs given here are usually new. We will also formulate general semi-classical asymptotics and Morse-type inequalities in a way which is more explicit, making use of a general notion of a model operator.

We shall start with some results about semi-classical asymptotics of spectra for second order elliptic self-adjoint operators. First we shall recall the classical example: quantum harmonic oscillator, which we will write in the form

$$H = -h\frac{d^2}{dx^2} + h^{-1}\omega^2 x^2 \ , \tag{0.1}$$

where $\omega > 0$, $h > 0$. Here h is supposed to be a small parameter ("Planck constant").

The operator (0.1) is considered in $L^2(\mathbf{R})$ and has discrete spectrum which consists of simple eigenvalues

$$\omega,\ 3\omega,\ 5\omega,\ \dots\ . \tag{0.2}$$

The calculation of the eigenvalues (0.2) can be found in any textbook on quantum mechanics (see e.g. [B-S] or [G-J]) where this operator is considered with different normalization (the operator (0.1) multiplied by h). The advantage of our normalization is that *the eigenvalues do not depend on h.*

Now let us consider a more general one-dimensional Schrödinger operator

$$H = -h\frac{d^2}{dx^2} + h^{-1}V(x)\ , \tag{0.3}$$

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where $V(x)$ is a smooth real-valued function such that $V(x) \to \infty$ as $|x| \to \infty$. This operator again has a discrete spectrum with the eigenvalues

$$\lambda_1(h) < \lambda_2(h) \leq \lambda_3(h) \leq \dots \tag{0.4}$$

such that $\lambda_j(h) \to \infty$ as $j \to \infty$ for every fixed $h > 0$ (see e.g. [B-S] or [R-S]). Here eigenvalues generally depend on h and the semi-classical asymptotics are asymptotics of the eigenvalues as $h \to 0$. It is easy to understand what can we expect of the eigenvalue $\lambda_j(h)$ with fixed j as $h \to 0$. By the general philosophy of quantum mechanics, up to exponentially small terms the eigenvalues should be defined by the behavior of the potential $V(x)$ in the deepest well (near the lowest minimum) or in all such wells (if there is more than one on the same minimal level), the interaction between wells (tunneling) being responsible for the exponentially small error terms only.

Suppose that all absolute minima are non-degenerate and are situated at the points $\bar{x}_1, \dots \bar{x}_N$. Denote the minimal value of V by V_{min}, so $V_{min} = V(\bar{x}_j)$, $j = 1, \dots, N$. Then

$$V'(\bar{x}_j) = 0 \text{ and } V''(\bar{x}_j) = 2\omega_j^2, \text{ where } \omega_j > 0$$

for all j. Let us assume for simplicity that $V_{min} = 0$ (the general case is reduced to this by adding a constant to the operator H which leads to the corresponding translation of the spectrum). Then we can expect that for small values of h the spectrum of H on any finite interval of the spectral parameter line (energy line) will be close to the spectrum of the orthogonal direct sum of harmonic oscillators

$$K = \bigoplus_{j=1}^{N} K_j \ , \tag{0.5}$$

where

$$K_j = -\frac{d^2}{dx^2} + \omega_j^2 x^2 \ . \tag{0.6}$$

The spectrum of K obviously coincides with the set

$$(2k_j + 1)\omega_j; \ j = 1, \dots, N, \ k_j = 0, 1, \dots \ . \tag{0.7}$$

Let us list all elements of this discrete set in the increasing order:

$$\bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \bar{\lambda}_3 \leq \dots \ , \tag{0.8}$$

each eigenvalue repeated according to its multiplicity. Then the simplest expected result is that for any fixed j

$$\lambda_j(h) \to \bar{\lambda}_j \text{ as } h \to 0 \ , \tag{0.9}$$

where $\lambda_j(h)$ are the eigenvalues from the list (0.4). This can be rigorously proved (see e.g. [C-F-K-S]).

Let us consider the Schrödinger operator (0.3) with the smooth potential $V(x)$ which is periodic with the period 1, i.e. $V(x+1) = V(x),\ x \in \mathbf{R}$. Again suppose that all the absolute minima are non-degenerate and $V_{min} = 0$. But all of them will be periodically repeated, so let us take only those of them which lie in a fundamental domain, say $[0, 1)$, and denote them again $\bar{x}_1, \dots, \bar{x}_N$. As before suppose that $V_{min} = 0$. Let us form the model operator K by the same formulas (0.5)-(0.6). What can we say about the spectrum of H?

It is well known that the spectrum of the one-dimensional periodic Schrödinger operator H is a union of "energy bands"

$$[a_1, b_1],\ [a_2, b_2],\ [a_3, b_3], \dots \ , \tag{0.10}$$

where $a_j = a_j(h)$, $b_j = b_j(h)$ and

$$a_1 < b_1 \le a_2 < b_2 \le a_3 < b_3 \le \dots \ , \tag{0.11}$$

i.e. the bands can not overlap but can join by the ends. There is a way to distinguish between different bands even if they join: the numbers a_{2k-1} and b_{2k} $(k = 1, 2, \dots)$ are eigenvalues of H with periodic conditions $\psi(x+1) = \psi(x)$ and the numbers b_{2k-1} and a_{2k} are eigenvalues of H with anti-periodic conditions $\psi(x+1) = -\psi(x)$. It occurs that energy bands behave as atoms in the semi-classical limit: if we use the same notations (0.9) for the eigenvalues of the model operator K then

$$a_j(h) \to \bar{\lambda}_j \text{ and } b_j(h) \to \bar{\lambda}_j \quad \text{as } h \to 0 \ . \tag{0.12}$$

(see [H2]). This means that each band shrinks to the corresponding eigenvalue of the model operator. In particular some gaps between bands may also shrink (the bands which correspond to the same multiple eigenvalue of K) while other gaps do not shrink but have a positive limiting length.

The same picture holds for the multidimensional Schrödinger operator in $\mathbf{R}^n$ with a (smooth) potential which is periodic with respect to a lattice (e.g. $\mathbf{Z}^n$) except the bands can overlap and we should use multidimensional quantum oscillators to form the model operator ([H2]).

2. Now let us turn to the case when we have periodicity with respect to a non-commutative group. We consider periodic self-adjoint second order elliptic differential operators in vector bundles on regular covering manifolds with a compact base. Here the periodicity means the invariance under a free action of a discrete group (generally non-commutative). The corresponding transformations of the manifold are usually called deck transformations.

The operators which we study have the form

$$H = -hA + B + h^{-1}V \ , \tag{0.13}$$

where $h > 0$ is a small parameter ("Planck constant"), $(-A)$ is a second order operator with a positive principal symbol (e.g. the Laplacian $dd^* + d^*d$ on differential forms), B

is a matrix zero order term and V is a non-negative zero order operator ("potential"). We require for simplicity that the (matrix) potential $V(x)$ vanishes at x as soon as one of its eigenvalues vanishes there. Moreover we require that a non-degeneracy condition is fulfilled at all the points x where $V(x)$ vanishes.

The operators (0.13) differ from the usual objects of the semi-classical study by a normalization factor h^{-1} which makes the formulations of the semi-classical asymptotics for the lower part of the spectrum more convenient.

The first result of this paper states that in the semi-classical limit the spectrum concentrates to the spectrum of a model operator which is a direct sum of harmonic-oscillator-type operators appearing near the critical points, i.e. the points where the spectrum of the potential contains 0. This shows an interesting feature of the spectrum: for small values of h the spectrum develops relatively large gaps, whereas some gaps which can appear near the eigenvalues of the model operator, actually shrink.

The next question that we address is how to measure the part of the spectrum near an eigenvalue of the model operator. It occurs that this part has the von Neumann dimension which is precisely equal to the multiplicity of the eigenvalue. Heuristically this means that for our purpose a good semi-classical approximation of the operator is provided by the direct sum of disjoint potential wells, where the operator can be replaced by the oscillator-type quadratic operator. Note that the number of these wells in our case is generally infinite (though they periodically repeat a finite number of wells which lie in a fundamental domain).

The operators (0.13) generalize the ones which appear when the Witten deformation, associated with a Morse function, is applied to the Laplacian on exterior p-forms on a compact riemannian manifold. The simplest Morse inequalities appear then as the fact that the dimension of the spectral subspace of the deformed Laplacian corresponding to a neighborhood of 0 on the spectral line (which is equal to the multiplicity of the zero eigenvalue of the model operator) is greater or equal than the dimension of the space of the zero modes for the same operator. The space of the zero modes is isomorphic to the space of the harmonic forms, so its dimension is the Betti number b_p. A non-trivial calculation which was done by Witten shows that the multiplicity of the zero eigenvalue of the model operator is equal to the number m_p of the critical points with the index p. Applying our general theorem to (non-compact) regular covering spaces and using von Neumann dimension instead of the usual one, we obtain more general L^2 Morse inequalities.

There is a number of other geometrical situations where the general theorem on semi-classical asymptotics can be applied. Here we give only one more example: L^2 version of the Novikov inequalities for vector fields. Other applications will be considered in a subsequent paper.

3. Let us give some additional history and references.

We refer the reader to [M] about the classical Morse inequalities. The L^2 Morse inequalities were first proved in [N-S1] by topological methods.

The E.Witten proof of the classical Morse inequalities appeared in [W1] where the Witten deformation was applied to the Laplacian on a compact riemannian manifold (see also [H], [H-S2], [Si2], [C-F-K-S] for a rigorous treatment). E.Witten used the large parameter T which is inverse to the small parameter h which we use in this paper.

Other results of [W1] (the Witten's proofs of the Poincaré-Hopf formula for vector fields and the Atiyah-Bott formula for the Hirzebruch signature in presence of a Killing vector field) were explained and developed in [E-W].

Later E.Witten [W2] applied his method to obtain holomorphic Morse inequalities. Another (asymptotic) version of holomorphic Morse inequalities was given by J.-P.Demailly ([D1], [D2], [D3], [Siu1]) who actually used oscillator type model operators to calculate the asymptotic of the heat kernel on $E^k \otimes F$ (here E is a holomorphic line bundle, F an arbitrary holomorphic vector bundle on the same complex manyfold) as $k \to \infty$. An application of these inequalities is a strong version of the Grauert-Riemenschneider conjecture which gives a description of the Moishezon varieties ([D4]). Y.-T.Siu ([Siu2]) used the Demailly inequalities to establish some purely analytic inequalities. J.-M.Bismut [B2] gave another proof of the Demailly's inequalitites. E.Getzler extended them to the case when the line bundle E is replaced by a holomorphic bundle of arbitrary rank ([G1]), and also to strictly pseudoconvex CR manifolds ([G2]).

J.-M. Bismut [B1] applied the Witten method and probability technique to give a new proof of the degenerate Morse-Bott inequalities. Two other analytic proofs of these inequalities were provided by B.Helffer and J.Sjöstrand [H-S3] and B.Helffer [H3].

A.V.Pazhitnov [P] used the Witten's method to prove some of the Morse-Novikov inequalities where the Morse function is replaced by a closed 1-form (these inequalities appeared in a paper of S.P.Novikov [No] and in an important particular case were improved by M.S.Farber [F]). S.P.Novikov (see [N-S1], Appendix) used the method to establish Morse inequalities for vector fields on compact manifolds (the detailed proofs and refined formulations of these inequalities can be found in [S7]).

The combinatorial version of the Witten method was developed by R.Forman in [F1], [F2].

It is quite natural to try to extend the Witten method to obtain Morse type inequalities on non-compact manifolds. This requires an additional structure (an appropriate trace in the context of von Neumann algebras or without them). To this end J.Roe [R] used a trace connected with a regular exhaustion on a Riemannian manifold of bounded geometry. In particular his result extends a result of M.Tsatsulin [Ts] who proved the Morse inequalities for almost periodic Morse functions on regular coverings of compact manifolds with an abelian group of automorphisms.

Other important applications of the Witten method are extensions and new proofs of the J.Cheeger and W.Müller result connecting the analytic Ray-Singer torsion with the combinatorial Reidemeister torsion – see [B-Z1], [B-Z2], [B-F-K1], [B-F-K2], [B-F-K-M]. In particular, the existence of a gap separating the spectrum near zero from the high energy spectrum for the Witten deformation of the Laplacian plays an important role in the proof of the coincidence of the analytic and combinatorial torsions (see [B-Z1], [B-Z2]); in L^2 case this existence was established in [B-F-K-M].

Periodic Schrödinger operators on $\mathbf{R}^n$ are a simplest particular case of the operators that we study here. For the one-dimensional Schrödinger operator with a periodic potential on $\mathbf{R}^1$ our qualitative result is equivalent to the above mentioned fact that the number of the energy bands which appear near an eigenvalue of the model operator is exactly equal to the multiplicity of this eigenvalue (this result is proved in [H2]). About further results on the periodic Schrödinger operators see [H2], [O], [Si2-Si6] and references there.

Both the semi-classical asymptotics found here and their applications to the Morse inequalities follow (in spirit and in some elements of the technique) the exposition given by B.Simon in [Si2] (see also Chapter 11 of the book [C-F-K-S]). However some important new elements have to be introduced to adapt the technique to the periodic case. This is first of all the use of the von Neumann algebra technique. Namely, we use the von Neumann Γ-dimension and the corresponding spectrum distribution function which is a generalization of the integrated density of states appearing in the solid state physics (see [B, B-L-T, B-S, S2, S3]). This spectrum distribution function proved to be useful in many aspects of analysis, spectral theory and topology on non-simply connected manifolds (see e.g. [C-M, C-G1, C-G2, Do, E, E-S, G-S, L, L-L, N-S1, N-S2, S6]). It is probably possible to extend deeper results of Helffer and Sjöstrand [H-S1, H-S2] (including their elaboration on the Witten's instanton approximation) to the periodic case but this was unnecessary for the applications to Morse inequalities that we kept in mind.

4.. I started this work after stimulating discussions with J.-M. Bismut and I am very grateful to him for these discussions. I am also grateful to B.Helffer, L.Hörmander, J.Roe and B.Simon for their bibliographical comments.

5. This paper is dedicated to the memory of F.A.Berezin who was one of the finest mathematicians and human beings that I knew. I always remember that I learned from him most of the mathematics that is used in this paper.

1. Preliminaries and main results.

1. Suppose that M is a C^∞ manifold, $\dim_{\mathbf{R}} M = n$, with a free C^∞ action of a discrete group Γ so that $X = M/\Gamma$ is compact. A natural example of this situation appears when M is the universal covering of a compact non-simply connected C^∞ manifold X and $\Gamma = \pi_1(X)$ is the fundamental group of X which acts on M by the deck transformations. Generally M can be any regular covering of X corresponding to a normal subgroup of $\pi_1(X)$ so that Γ is the quotient group.

The group Γ acts naturally on functions and on all tensor spaces on M. In particular Γ acts on differential forms. More generally consider a complex vector Γ-bundle E on M, i.e. a C^∞ complex vector bundle on M with an action of Γ in E which covers the action of Γ on the base X i.e. for any $\gamma \in \Gamma$ and any $x \in X$ we should have a linear map $\hat{\gamma}_x : E_x \to E_{\gamma x}$. Then we have a natural action of Γ in the space $C^\infty(M, E)$ of all smooth sections of E over M; namely for every $\gamma \in \Gamma$ we can form a linear map $L_\gamma : C^\infty(M, E) \to C^\infty(M, E)$ by

$$L_\gamma f(x) = \hat{\gamma}_{\gamma^{-1}x} f(\gamma^{-1}x).$$

Then the correspondence $\gamma \mapsto L_\gamma$ becomes a representation of Γ in $C^\infty(M, E)$.

We shall denote by k the (complex) dimension of the fiber of E.

Now suppose that we have chosen a Γ-invariant 1-density (or volume element) on M (it will usually be given as a riemannian volume element of a Γ-invariant riemannian metric on M). Suppose also that we have a Γ-invariant hermitian structure in fibers of E. Then we can form a Hilbert space $L^2(M,E)$ of all square-integrable sections of E over M and Γ acts in this Hilbert space by unitary transformations.

Consider a Γ-invariant (or, in other words, Γ-periodic) linear elliptic differential operator

$$H:\ C^\infty(M,E) \longrightarrow C^\infty(M,E). \tag{1.1}$$

Here the Γ-invariance means that H commutes with the action of Γ on sections. A formally adjoint operator H^* is well defined and is also a Γ-invariant linear elliptic differential operator. We shall usually suppose that H is formally self-adjoint i.e. $H^* = H$. In this case H naturally defines a self-adjoint operator in $L^2(M,E)$ (see e.g. [S4]) which will be also denoted by H. This operator commutes with the action of Γ in $L^2(M,E)$. We will usually refer to such self-adjoint operators as hamiltonians.

A natural example of a hamiltonian is the Laplacian

$$\Delta_p = -(dd^* + d^*d):\ \Lambda^p(M) \longrightarrow \Lambda^p(M)$$

defined on the space $\Lambda^p(M)$ of all C^∞ differential forms of degree p on M. Here $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ is the exterior differential on M, d^* its formal adjoint. The most important hamiltonians that we will consider are obtained from the Laplacian Δ_p or from the direct sum of such Laplacians by adding to them some Γ-invariant 0-order operators.

The main object of our study will be the asymptotic behavior of the spectrum and the spectrum distribution function of a hamiltonian which depends on a small parameter $h > 0$ ("Planck constant") and has the form

$$H = -hA + B + h^{-1}V\ . \tag{1.2}$$

Here A is a second order self-adjoint elliptic differential operator with a negative principal symbol (e.g. Δ_p above), so that $-A$ is semi-bounded from below; B is a zero order self-adjoint operator i.e. an hermitian endomorphism of the bundle E which is locally just a multiplication operator by an hermitian $k \times k$ matrix function; V is again a self-adjoint 0-order operator. For the sake of simplicity we suppose all coefficients to be in C^∞, though in fact C^3 is sufficient almost everywhere.

Since we suppose H to be Γ-periodic, both $B(x)$ and $V(x)$ should be Γ-periodic too.

The simplest operator of the form (1.2) (though not a periodic one) is the (divided by h) one-dimensional harmonic oscillator

$$H = -h\frac{d^2}{dx^2} + h^{-1}\omega^2x^2, \tag{1.3}$$

where $\omega > 0$. It is considered in $L^2(\mathbf{R}^1)$ and has the eigenvalues

$$\{(2k+1)\omega;\ k = 0, 1, 2, \ldots\} \tag{1.4}$$

(see e.g. [B-S] or [G-J]). So these eigenvalues do not depend on h and this is why we chose to divide the standard harmonic oscillator by h. This operator will be the simplest model operator for us. In general case we will consider the limit behavior of the spectrum of the operator H on a fixed finite interval $[-R, R]$ where $R > 0$ is an arbitrary constant. We will see that under appropriate conditions the spectrum will have reasonable asymptotic behavior (even for the periodic case where the spectrum is usually continuous).

Let us formulate the conditions on the components of the hamiltonian in (1.2).

Our main condition is as follows:

(C). *$V(x) \geq 0$ for all $x \in M$. If at a point $\bar{x} \in M$ the matrix $V(\bar{x})$ degenerates then $V(\bar{x}) = 0$ and*

$$V(x) \geq c|x - \bar{x}|^2 I \text{ in a neighbourhood of } \bar{x}. \tag{1.5}$$

Here I is the identity morphism of the corresponding fiber and the inequality is understood as the inequality of quadratic forms.

It follows that all points of degeneracy are isolated. Denote $S_{\min}$ the set of all degeneracy points for V. This is a discrete Γ-periodic subset in M. In particular any fundamental domain F of Γ on M contains only a finite number of points in $S_{\min}$. Let us denote these points $\bar{x}_1, \ldots, \bar{x}_N$.

Denote $\bar{B}_j = B(\bar{x}_j)$, $j = 1, \ldots, N$, so $\bar{B}_j$ is an endomorphism of the fiber of the bundle E over the point $\bar{x}_j$.

Now we want to form a model operator which will have a relatively simple form: a matrix harmonic oscillator which is as close to H near $\bar{x}_j$ as possible.

Suppose that we have chosen local coordinates $x^1, \ldots, x^n$ in a neighborhood U of $\bar{x}_j$ and a trivialization of the vector bundle E near $\bar{x}_j$. Let us assume that $\bar{x}_j$ becomes 0 in these local coordinates and that the hermitian form in the fibers of E becomes standard in the chosen trivialization. The given volume element in these coordinates locally has the form $\sqrt{g(x)}dx$ where $g \in C^\infty$ and $g > 0$ near 0 (for the riemannian volume g will be equal to $\det[g_{rs}]$ where g_{rs} are the components of the metric tensor). Being self-adjoint the operator A should locally have the form

$$A = A^{(2)} + A^{(1)} + A^{(0)},$$

where $A^{(s)}$ is an operator of order s, $s = 0, 1, 2$,

$$A^{(2)} = \sum_{1 \leq r,s \leq k} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} \sqrt{g} A_{rs}(x) \frac{\partial}{\partial x^s}, \quad A_{rs}^* = A_{sr}; \tag{1.6}$$

$$A^{(1)} = \sum_{1 \leq r \leq n} A_r(x) \frac{\partial}{\partial x^r}, \tag{1.6'}$$

where A_{rs} and A_r are $k \times k$ smooth matrix functions defined in a neighborhood of 0; $A^{(0)}$ is just a multiplication by a smooth matrix function $A^{(0)}(x)$.

The principal symbol of $-A$ in the chosen coordinates and trivialization is the matrix function on $U \times \mathbf{R}^n$

$$a^{(2)}(x,\xi) = \sum_{1\le r,s\le k} \xi_r \xi_s A_{rs}(x) \ . \tag{1.7}$$

In fact it is well defined as a function on the cotangent bundle T^*M (its value at a point $(x,\xi) \in T_x^*M$ is an endomorphism of the fiber E_x over x). For the self-adjoint operator $-A$ its ellipticity and semi-boundedness from below mean that the matrix $a^{(2)}(x,\xi)$ is positive definite for all (x,ξ) with $\xi \neq 0$ at all points $x \in M$.

An important example of such an operator A is the Laplacian on the differential p-forms on a riemannian manifold. For example if we consider the Laplacian on scalar functions, then we should take $A_{rs} = g^{rs},\ A_r = 0$.

Denote

$$A^{(2)}_{flat} = \sum_{1\le r,s\le n} A_{rs}(0) \frac{\partial^2}{\partial x^r \partial x^s}, \tag{1.8}$$

so $A^{(2)}_{flat}$ is the homogeneous second order differential operator with constant coefficients which are obtained by freezing the coefficients of the principal part of A at the point $\bar{x}_j$.

The "potential" $V(x)$ in the chosen local coordinates near $\bar{x}_j$ can be written in the form $V = V^{(2)} + R$, where

$$V^{(2)}(x) = \frac{1}{2} \sum_{1\le r,s\le k} \frac{\partial^2 V}{\partial x^r \partial x^s}(0) x^r x^s; \qquad R(x) = O(|x|^3) \text{ as } x \to 0. \tag{1.9}$$

This means that $V^{(2)}$ is the quadratic part of V near $\bar{x}_j$ (constant and linear parts obviously vanish).

It will be convenient for us to use local coordinates near $\bar{x}_j$ such that $\bar{x}_j$ becomes 0 in these coordinates, and $g(0) = 1$, i.e. the riemannian volume element at the point $\bar{x}_j$ coincides with the volume element given by the chosen local coordinates. We shall fix the choice of such local coordinates near each point $\bar{x}_j$ and refer to them as the *canonical coordinates*. In particular we can take the usual geodesic canonical coordinates, but not necessarily. If we shall need local coordinates near $\gamma\bar{x}_j$ then we can always consider the coordinates obtained from the chosen canonical coordinates at $\bar{x}_j$ by the translation given by the action of γ. Similarly we shall fix a trivialization of the bundle E near $\bar{x}_j$ such that the hermitian metric becomes trivial in this trivialization, and we will refer to this trivialization as a *canonical trivialization.* The canonical trivialization near $\gamma\bar{x}_j$ will be always obtained by translation given by the action of γ in E.

Now we can form a model operator K_j for H near $\bar{x}_j$:

$$K_j = -A^{(2)}_{flat} + \bar{B}_j + V^{(2)}(x), \tag{1.10}$$

where both $A^{(2)}_{flat}$ and $V^{(2)}(x)$ are taken in canonical coordinates and in a canonical trivialization. This operator is considered in $L^2(\mathbf{R}^n) \bigotimes \mathbf{C}^k$. It is a matrix analog of the quantum harmonic oscillator and it has a discrete spectrum which is semi-bounded from below. Since we fixed the volume element given by the riemannian metric it is easy to check that the spectrum of K_j does not depend on the choice of the canonical coordinates and the canonical trivialization.

Later on we will also need the dilated operator

$$K_j(h) = -hA^{(2)}_{flat} + \bar{B}_j + h^{-1}V^{(2)}(x), \tag{1.10$'$}$$

It is easy to see that its spectrum does not depend on the parameter $h > 0$, hence $\mathrm{spec}(K_j(h)) = \mathrm{spec}(K_j)$ for all $h > 0$. (Here and later for any self-adjoint operator H we will denote its spectrum by $\mathrm{spec}(H)$.)

Define at last the total model operator

$$K = \bigoplus_{1\le j\le N} K_j, \tag{1.11}$$

which is a self-adjoint operator acting in the orthogonal direct sum
$\bigoplus_{1\le j\le N}(L^2(\mathbf{R}^n) \bigotimes \mathbf{C}^k)$.

If H is a self-adjoint operator then we shall use the truncated spectrum

$$\mathrm{spec}_R(H) = \mathrm{spec}(H) \cap [-R, R].$$

For any two compact subsets $S_1, S_2 \subset \mathbf{R}$ denote the Hausdorff distance between them by $d(S_1, S_2)$:

$$d(S_1, S_2) = \max\{\max_{x\in S_1}\min_{y\in S_2}\{|x-y|\},\ \ \max_{x\in S_2}\min_{y\in S_1}\{|x-y|\}\}.$$

Now we can formulate our first claim:

Theorem 1.1. *Suppose that H is given by* (1.2) *and the condition* (C) *is satisfied. Then the spectrum of the operator H concentrates near the spectrum of the model operator K as $h \to 0$. More precisely, for every $R > 0$ with $\pm R \notin \mathrm{spec}(K)$ there exist $h_0 > 0$ and $C > 0$ such that*

$$d(\mathrm{spec}_R(H), \mathrm{spec}_R(K)) \le Ch^{1/5} \ \text{ if } \ h \in (0, h_0). \tag{1.12}$$

Remark 1.2. Since the spectrum of K is discrete, the claim actually means that for small h any bounded part of the spectrum of H is located in a $Ch^{1/5}$-neighborhood of the eigenvalues of K and, moreover, for any eigenvalue of the model operator K the corresponding part of the spectrum of H is not empty.

In fact we shall give a more precise information about the location of the spectrum of H. Namely, suppose that

$$\mu_1 < \mu_2 < \mu_3 \dots \tag{1.13}$$

is the set of all eigenvalues of K. (There might be multiple ones among them). Then we shall actually prove that for every $R > 0$ and $\kappa \in (0, 1/2)$ there exist h_0 and $C > 0$ such that for every $h \in (0, h_0)$

$$\operatorname{spec}_R(H) \subset \bigcup_{j=1}^{\infty} (\mu_j - Ch^{1/5}, \mu_j + Ch^{\kappa}) \,. \tag{1.14}$$

So the right ends of the "energy bands" can be estimated better than the left ones.

Even more precise information about the right ends can be obtained if we assume that the operator H is *flat* near the points $\bar{x}_j$, i.e. if in the canonical coordinates and canonical trivialization in a neighborhood of $\bar{x}_j$ the riemannian metric is flat and besides we have in this neighborhood precisely $H = K_j(h)$ for all $j = 1, \dots N$. Then for every $R > 0$ and every $\varepsilon > 0$ there exist $h_0 > 0$ and $C > 0$ such that

$$\operatorname{spec}_R(H) \subset \bigcup_{j=1}^{\infty} (\mu_j - Ch^{1-\varepsilon}, \mu_j + C\exp(-C^{-1}h^{-1+\varepsilon})) \,. \tag{1.15}$$

Remark 1.3. For the periodic Schrödinger operator in $\mathbf{R}^n$ the localization of the spectrum given in Theorem 1.1 can be made much more precise by considering energy bands and investigating asymptotics of eigenvalues corresponding to any fixed value of the quasimomentum. Also more precise results on the length of the localization interval can be obtained if for any energy well we consider the operator H itself (without taking quadratic parts etc.) in this energy well with the Dirichlet boundary conditions as a model operator. In particular, it can be established in this way that the first energy band for the periodic Schrödinger operator in $\mathbf{R}^n$ is exponentially small. But the place of the exponential localization is difficult to find explicitly.

Also more subtle results on interaction between different wells can be established - see [H2, H-S1, H-S2, O, Si1-Si5] for more details.

Remark 1.4. An explicit calculation of the eigenvalues of the model operators ("matrix oscillators") is a very interesting problem. There are several important cases when it can be solved. It is easy to see that in case when $A^{(2)}_{flat}$ and $V^{(2)}$ are in fact scalar matrices (i.e. the identity matrices multiplied by a fixed differential operator or a fixed quadratic form in x-variables respectively), the calculation of the eigenvalues reduces to the calculation of the eigenvalues of matrices. This in turn may occur to be a difficult problem, especially because we usually have to deal not with an individual matrix but with a series of matrices i.e. a matrix depending on an integer parameter. In two geometrically important particular cases the appearing problems were solved by E.Witten ([W1], see also [H-S2]) and S.Novikov (see [N-S1], Appendix, and also [S7]) for the sake of proving the classical Morse inequalities and Morse-type inequalities for vector fields.

A simplest example of a more complicated nature is the 1-dimensional matrix Schrödinger operator

$$K = -A\frac{d^2}{dx^2} + Bx^2,$$

where A and B are constant (non-commuting) positive definite hermitian matrices. As far as I know the eigenvalues of such an operator were never calculated in the general case (even for 2×2 matrices).

In case of relatively small values of k and n the calculation of first eigenvalues can be done by computer calculations.

2. Now we shall describe a quantitative characteristic of the part of the spectrum of the operator H situated in a small neighborhood of an eigenvalue of K. We shall do this in terms of von Neumann Γ-trace and Γ-dimension. We shall very briefly describe the definition of the Γ-dimension and the Γ-trace. For more details we refer the reader to [At], [C] and textbooks on von Neumann algebras (e.g. [D], [N], [T]).

We shall denote the Γ-dimension by $\dim_\Gamma$. It is defined on the set of all (projective) Hilbert Γ-modules and takes values in $[0,\infty]$. The simplest Hilbert Γ-module is given by a left regular representation of Γ: it is the Hilbert space $L^2\Gamma$ consisting of all complex-valued L^2-functions on Γ. The group Γ acts unitarily on $L^2\Gamma$ by $\gamma \mapsto L_\gamma$ where L_γ is defined as follows:

$$L_\gamma f(x) = f(\gamma^{-1}x),\ x \in \Gamma; \quad f \in L^2\Gamma.$$

By definition $\dim_\Gamma L^2\Gamma = 1$.

For any (complex) Hilbert space $\mathcal{H}$ define a free Hilbert Γ-module $L^2\Gamma \bigotimes \mathcal{H}$. Its Γ-dimension equals $\dim_{\mathbf{C}} \mathcal{H}$. The action of Γ in $L^2\Gamma \bigotimes \mathcal{H}$ is defined by $\gamma \mapsto L_\gamma \otimes I$.

A general Hilbert Γ-module is a closed Γ-invariant subspace in a free Hilbert Γ-module. It would be natural to call such subspaces *projective* Hilbert modules, but the word " projective" is usually omitted, so only projective Hilbert modules are considered.

For any Hilbert space $\mathcal{H}$ denote by $\mathcal{A}_\Gamma$ a von Neumann algebra which consists of all bounded linear operators in $L^2\Gamma \bigotimes \mathcal{H}$ which commute with the action of Γ there. This algebra is in fact generated by the operators of the form $R_\gamma \otimes B,\ B \in \mathcal{B}(\mathcal{H}),\ \gamma \in \Gamma$, where $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators in $\mathcal{H}$, R_γ is the operator of the right translation in $L^2\Gamma$ i.e.

$$R_\gamma f(x) = f(x\gamma),\ x \in \Gamma; \quad f \in L^2\Gamma.$$

This means that the algebra $\mathcal{A}_\Gamma$ is the weak closure of all finite linear combinations of the operators of the form $R_\gamma \otimes B$. So in fact $\mathcal{A}_\Gamma$ is a tensor product (in the sense of von Neumann algebras) of $\mathcal{R}_\Gamma$ and $\mathcal{B}(\mathcal{H})$ where $\mathcal{R}_\Gamma$ is the von Neumann algebra generated by the operators R_γ in $L^2\Gamma$ (it consists in fact from all operators in $L^2\Gamma$ which commute with all operators $L_\gamma,\ \gamma \in \Gamma$).

There is a natural trace on $\mathcal{R}_\Gamma$. It is denoted by tr_Γ and defined as the diagonal matrix element (all of them are equal) in the δ-functions basis. For example we can define it by

$$\mathrm{tr}_\Gamma S = (S\delta_e, \delta_e), \quad S \in \mathcal{R}_\Gamma,$$

where e is the neutral element of Γ, $\delta_e \in L^2\Gamma$ is the "Dirac delta-function" at e, i.e. $\delta_e(x) = 1$ if $x = e$ and 0 otherwise. There is also a natural trace on $\mathcal{A}_\Gamma$ too: $\mathrm{Tr}_\Gamma = \mathrm{tr}_\Gamma \otimes \mathrm{Tr}$ where Tr is the usual trace on $\mathcal{B}(\mathcal{H})$.

Now for any Hilbert Γ-module which is a Γ-invariant subspace L in $L^2\Gamma \bigotimes \mathcal{H}$, its Γ-dimension is defined by the natural formula

$$\dim_\Gamma L = \mathrm{Tr}_\Gamma P_L,$$

where P_L is the orthogonal projection on L in $L^2\Gamma \bigotimes \mathcal{H}$.

For any self-adjoint operator H and any Borel subset $J \subset \mathbf{R}$ denote by $E(J; H)$ the spectral projection of H corresponding to the subset J. We will sometimes omit H and write $E(J)$ instead of $E(J; H)$ if H is obvious from the context. In particular we shall use the projection $E_\lambda = E_\lambda(H) = E((-\infty, \lambda]; H)$.

For any self-adjoint operator H commuting with the action of Γ in a Hilbert Γ-module define the von Neumann spectrum distribution function of this operator $N_\Gamma(\cdot; H) : \mathbf{R} \longrightarrow [0, \infty]$ as

$$N_\Gamma(\lambda; H) = \mathrm{Tr}_\Gamma E_\lambda(H).$$

It is a non-decreasing function on $\mathbf{R}$. If it is finite for all $\lambda \in \mathbf{R}$, then $\mathrm{spec}(H)$ coincides with the set of all points of growth of this function, namely:

$$\mathrm{spec}(H) = \{\lambda |\, N_\Gamma(\lambda + \varepsilon; H) - N_\Gamma(\lambda - \varepsilon; H) > 0 \ \text{ for every } \ \varepsilon > 0.\} \tag{1.16}$$

More generally, for any Borel subset $J \subset \mathbf{R}$ define $N_\Gamma(J; H) = \mathrm{Tr}_\Gamma E(J; H)$.

For operators H which have a discrete spectrum and are semi-bounded from below, we shall also use the usual spectrum distribution function $N(\lambda; H) = \mathrm{Tr}\, E_\lambda$ (the number of eigenvalues $\leq \lambda$).

We shall usually deal with the Γ-modules which appear as closed Γ-invariant subspaces of the (free) Hilbert Γ-module $L^2(M, E)$, which is the space of L^2-sections of an hermitian Γ-bundle E on a regular covering M of a compact riemannian manifold $X = M/\Gamma$. Denote by F the fundamental domain of the action of Γ. Suppose that an operator in $L^2(M, E)$ has a finite Γ-trace and a continuous Schwartz kernel $K_A(x, y)$ i.e. $K_A(x, y) : E_x \to E_y$ is a linear map of fibers and the operator is given by

$$Au(x) = \int_M K_A(x, y) u(y) d\mu(y), \quad u \in L^2(M, E),$$

where $d\mu$ is the (Γ-invariant) riemannian volume element associated with the chosen Γ-invariant riemannian metric on M. Then

$$\mathrm{Tr}_\Gamma A = \int_F \mathrm{tr} K_A(x, x) d\mu(x) \tag{1.17}$$

(see e.g. [At]). Here $\mathrm{tr} K_A(x,x)$ means the usual matrix trace of $K_A(x,x)$ which is an endomorphism of the fiber E_x.

Returning to our analytical context, note that the von Neumann spectrum distribution function is well defined and finite for any operator H of the form (1.2) ([At]).

Now we can formulate the main quantitative result which supplements Theorem 1.1.

Theorem 1.5. *Let us assume that the condition (C) is satisfied. Then for any $R > 0$ and $\kappa \in (0, 1/2)$ there exist $C > 0,\ h_0 > 0$ such that for all $\lambda \in [-R, R]$ and $h \in (0, h_0)$*

$$N(\lambda - Ch^{\kappa}; K) \le N_{\Gamma}(\lambda; H) \le N(\lambda + Ch^{1/5}; K)\ . \tag{1.18}$$

So the von Neumann spectrum distribution function of H becomes very close to the usual spectrum distribution function of the model operator K on any finite interval as $h \to 0$. But note that in fact Theorem 1.5 gives much stronger information, providing the existence of large gaps in the spectrum. Indeed, since the function $N(\cdot; K)$ is constant on all intervals

$$(-\infty, \mu_1),\ (\mu_1, \mu_2),\ (\mu_2, \mu_3), \dots\ ,$$

we immediately obtain the following

Corollary 1.6. *For any $R > 0$ and $\kappa \in (0, 1/2)$ there exist $C > 0,\ h_0 > 0$ such that for all $h \in (0, h_0)$*

$$N_{\Gamma}(\lambda; H) = N(\lambda; K) \text{ if } \lambda \in [-R, R], \text{ and } \lambda \notin \bigcup_{j=1}^{\infty} (\mu_j - Ch^{1/5}, \mu_j + Ch^{\kappa})\ . \tag{1.19}$$

Note in particular that together with (1.16) this Corollary implies Theorem 1.1 and the more precise estimate (1.14).

Remark 1.7. If H is flat in a neighborhood of $\bar{x}_j$ for all $j = 1, \dots, N$ then instead of (1.18) we can prove the following more precise estimate:

$$N(\lambda - C\exp(-C^{-1}h^{-1+\varepsilon}); K) \le N_{\Gamma}(\lambda; H) \le N(\lambda + Ch^{1-\varepsilon}; K)\ , \tag{1.20}$$

where $\varepsilon > 0$. This will imply that we can upgrade (1.19) to

$$N_{\Gamma}(\lambda; H) = N(\lambda; K) \text{ if } \lambda \in [-R, R], \text{ and } \lambda \notin \bigcup_{j=1}^{\infty} (\mu_j - Ch^{1-\varepsilon}, \mu_j + C\exp(-C^{-1}h^{-1+\varepsilon}))\ . \tag{1.21}$$

Let us make it more explicit. Denote by

$$r_1,\quad r_2,\quad r_3,\quad \dots \tag{1.22}$$

the multiplicities of the eigenvalues (1.13). Then the von Neumann spectrum distribution function $N_\Gamma(\cdot; H)$ takes the following values on gaps in the spectrum:

$$0 \ \text{ on } \ (-\infty, \mu_1 - Ch^{1/5}),$$

$$r_1 \ \text{ on } \ (\mu_1 + Ch^\kappa, \mu_2 - Ch^{1/5}),$$
$$r_1 + r_2 \ \text{ on } \ (\mu_2 + Ch^\kappa, \mu_3 - Ch^{1/5}),$$
$$r_1 + r_2 + r_3 \ \text{ on } \ (\mu_3 + Ch^\kappa, \mu_4 - Ch^{1/5}),$$

etc. (Again we can replace here h^κ by $\exp(-C^{-1}h^{-1+\varepsilon})$ and $h^{1/5}$ by $h^{1-\varepsilon}$ if H is flat near all points $\bar{x}_j$.) This immediately leads to the following

Corollary 1.8. *For any $R > 0$ there exist $C > 0$ and $h_0 > 0$ such that for any $j = 1, 2, 3, \ldots$ with $\mu_j \in [-R, R]$ and any $h \in (0, h_0)$*

$$N_\Gamma((\mu_j - Ch^{1/5}, \mu_j + Ch^\kappa); H) = r_j \ , \tag{1.23}$$

where h^κ can be replaced by $\exp(-C^{-1}h^{-1+\varepsilon})$ and $h^{1/5}$ by $h^{1-\varepsilon}$ if H is flat near all points $\bar{x}_j$.

This statement means that the von Neumann dimension of the spectral subspace of the operator H, corresponding to the part of the spectra near an eigenvalue of the model operator, is exactly equal to the multiplicity of this eigenvalue.

Corollary 1.9. *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$*

$$\dim \operatorname{Ker} K \geq \dim_\Gamma \operatorname{Ker} H \ . \tag{1.24}$$

This Corollary is the base for the applications to Morse-type inequalities.

Note that the spectrum of H may be continuous. We have seen that for small values of h it develops relatively large gaps. However some small gaps may exist as well near the eigenvalues of the model operator K.

It is well known (see e.g. [S2, S3, B, B-L-T]) that for periodic (and even almost periodic) operators in $\mathbf{R}^n$ the von Neumann spectrum distribution function coincides with the integrated density of states which is defined as a limit of the usual spectrum distribution functions of the given operator in bounded domains, normalized by division by the volume of the domain where the operator is considered. Here the operator on a bounded domain should be considered with the Dirichlet (or any other elliptic) boundary condition, and the limit is taken when the domain blows up in a sufficiently regular way (e.g. homothetically). In the periodic case the integrated density of states can be written as $(\operatorname{vol} F)^{-1}\operatorname{Tr}_\Gamma E_\lambda(H)$, so it differs from the spectrum distribution function $N_\Gamma(\lambda; H)$ by a constant factor $(\operatorname{vol} F)^{-1}$ only.

For the one-dimensional periodic Schrödinger operator the standard theory of quasi-momentum and its connection with the integrated density of states (see e.g. [S3]) implies immediately that the statement of the Corollary 1.8 in this case is equivalent to the fact that there are exactly r_j spectral bands of the operator H in the $Ch^{1/5}$-neighborhood of μ_j. So some gaps may indeed become very small provided the model operator has multiple eigenvalues; in the meantime the length of other gaps does not tend to 0.

All these results about the one-dimensional periodic Schrödinger operator, similar multidimensional results and more precise information about the length of the first energy band can be obtained by investigating Bloch spectrum for varying values of the quasimomentum (see [Si5], [H2] and references there).

3. Now we shall turn to the applications of the semiclassical asymptotics given above.

Let $f : X \to \mathbf{R}$ be a Morse function on $X = M/\Gamma$ i.e. a C^∞ function which has only non-degenerate critical points. Denote $\mathrm{Cr}(f) = \{x |\, x \in X,\ df(x) = 0\}$, i.e. $\mathrm{Cr}(f)$ is the set of all critical points of f. All the critical points are isolated. For $x \in \mathrm{Cr}(f)$ denote $\mathrm{ind}(x)$ the index of the critical point x, i.e. the number of the negative eigenvalues of the Hessian $f''(x)$. So $\mathrm{ind}(x) \in \{1, \dots, n\}$ where $n = \dim_{\mathbf{R}} X$. Denote $m_p = m_p(f)$ the number of its critical points with the index equal to p.

Let us denote by $\mathcal{H}_p(M)$ the space of all L^2 harmonic p-forms on M i.e. smooth p-forms ω belonging to L^2 and satisfying the equation $\Delta_p\omega = 0$. The space $\mathcal{H}_p$ is a Hilbert Γ-module and we can introduce the L^2-Betti numbers $\bar{b}_p = \dim_\Gamma \mathcal{H}_p$.

The numbers $\bar{b}_p$ were introduced by M.Atiyah [At], and later J.Dodziuk [Do] proved that they are homotopy invariants of X (see also [G-S] for an alternative proof). These numbers are usually considered in the case when M is the universal covering of X and $\Gamma = \pi_1(X)$ is the fundamental group of X. The numbers $\bar{b}_p$ depend on the choice of the covering manifold M and their homotopy invariance in the general case means that they are homotopy invariants on the category of smooth Γ-manifolds with the compact quotient. They can be also defined combinatorially ([Do]).

We refer the reader to [M] for the classical Morse inequalities.

Now the Morse inequalities with the L^2 Betti numbers look as follows:

Theorem 1.10. (S.P.Novikov, M.A.Shubin [N-S1]) *The following inequalities are true:*

(1.25) $$\sum_{j=0}^{p}(-1)^{p-j}m_j \geq \sum_{j=0}^{p}(-1)^{p-j}\bar{b}_j; \quad p = 0, \dots, n.$$

In case $p = n$ this inequality becomes equality.

Note that the weaker inequalities

(1.26) $$m_p \geq \bar{b}_p; \quad p = 1, \dots, n;$$

follow from (1.25) by adding two of the inequalities in (1.25) corresponding to p and $p-1$.

A topological proof of this theorem was given in [N-S1]. An almost periodic version of this theorem was proved by M.Tsatsulin [Ts] in case of abelian coverings. We shall provide an analytic proof of the Theorem 1.10 by adjusting the Witten's proof [W1] to the context of the periodic situation and von Neumann dimensions.

4. Here we shall provide an L^2 version of Novikov inequalities for vector fields (see Appendix to [N-S1] for the original compact manifolds version). Suppose that we are given a vector field $\mathbf{v}$ on X or, equivalently, a Γ-invariant vector field on M (which we shall denote by the same letter). We shall suppose that all singular points of $\mathbf{v}$ (the points where it vanishes) are non-degenerate i.e. if in local coordinates $x = (x^1, \dots, x^n)$ the field has the components $v_1(x), \dots, v_n(x)$ and the singular point is $\bar{x}$ in these local coordinates then

$$\det[\partial v_i/\partial x_j](\bar{x}) \neq 0.$$

The index of the singular point, denoted $\operatorname{ind}(\bar{x}; \mathbf{v})$, is ± 1 depending on the sign of this determinant.

Denote by $m_\pm$ the number of the singular points of $\mathbf{v}$ with the index ± 1 respectively.

Suppose that a Γ-invariant riemannian metric on M is fixed and consider the 1-form ω corresponding to $\mathbf{v}$ by the isomorphism of tangent and cotangent bundles defined by the given metric. Let us define a deformed exterior differential as $d_h = d + h^{-1}\omega$ where ω is identified with the operator of the exterior multiplication by ω. Define also the corresponding Laplacian which we will normalize by additional factor h:

$$\Delta_\omega = h(d_h + d_h^*)^2. \tag{1.27}$$

An important feature of this Laplacian is that it does not generally preserve the degree of the form but it preserves the parity of this degree, so it maps the spaces $\Lambda^{ev}(M)$ and $\Lambda^{odd}(M)$ into themselves. Denote the corresponding restrictions of Δ_ω by $\Delta_\omega^\pm$. Denote

$$\bar{b}_\pm(h) = \dim \operatorname{Ker} \Delta_\omega^\pm, \tag{1.28}$$

$$\bar{b}_\pm = \limsup_{h \to 0} \bar{b}_\pm(h). \tag{1.29}$$

Theorem 1.11. *The following two inequalities are true:*

$$m_\pm \geq \bar{b}_\pm. \tag{1.30}$$

Also

$$m_+ - m_- = \bar{b}_+(h) - \bar{b}_-(h) = \chi(X) \quad \text{for all} \quad h > 0. \tag{1.31}$$

Here $\chi(X)$ is the Euler characteristics of X.

2. Estimate from below.

1. We shall start from some elementary remarks about spectra of model operators.

Recall the following standard notations:

$C_0^\infty(\mathbf{R}^n)$ is the space of all complex-valued C^∞ functions with compact support in $\mathbf{R}^n$;

$\mathcal{S}(\mathbf{R}^n)$ is the Schwartz space of all complex-valued C^∞ functions in $\mathbf{R}^n$ with all derivatives decaying faster than any power of $|x|$ as $|x| \to \infty$.

Consider a model operator which has the form

$$K = -A(D) + B + V(x), \tag{2.1}$$

where $D = i^{-1}\partial/\partial x$; $A(\xi)$ and $V(x)$ are homogeneous quadratic polynomials in $\xi = (\xi_1, \dots, \xi_n)$ and $x = (x_1, \dots, x_n)$ respectively, with $k \times k$ matrix coefficients, such that $-A(\xi)$ and $V(x)$ are positive definite hermitian matrices for all non-zero values of their arguments; B is a constant hermitian $k \times k$ matrix. This operator is considered in $L^2(\mathbf{R}^n) \bigotimes \mathbf{C}^k$ e.g. as a closure from the domain $C_0^\infty(\mathbf{R}^n) \bigotimes \mathbf{C}^k$.

Lemma 2.1. *The operator K is self-adjoint and semi-bounded from below. It has a discrete spectra with all eigenfunctions in $\mathcal{S}(\mathbf{R}^n) \bigotimes \mathbf{C}^k$.*

Moreover the eigenfunctions satisfy the following estimate:

$$|\psi(x)| \le C \exp(-ax^2), \tag{2.2}$$

where C and a are positive constants (depending on ψ).

Proof. All statements except the estimate (2.2) follow immediately from the theory of globally elliptic operators (see e.g. [S4], Chapter 4, or [H1]). The estimate (2.2) will be proved in the Appendix. For similar estimates for scalar Schrödinger operators see [A], [B-S] Ch.3, [Co], [C-T], [R-S] Ch.XIII.11 and [Si1]. Note that (2.2) is only needed for the proof of the refined estimates like (1.15), (1.20), (1.21) etc. and they are irrelevant for the Morse inequalities. □

Denote

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \tag{2.3}$$

the eigenvalues of K ($\lambda_m \to \infty$ as $m \to \infty$),

$$\psi_1(x), \quad \psi_2(x), \quad \psi_3(x), \dots \tag{2.3'}$$

the corresponding eigenfunctions (which are $\mathbf{C}^k$-valued Schwartz functions on $\mathbf{R}^n$ and are supposed to be normalized in $L^2(\mathbf{R}^n) \bigotimes \mathbf{C}^k$).

Let us add the small parameter $h > 0$ and consider the modified operator

$$K(h) = -hA(D) + B + h^{-1}V(x). \tag{2.4}$$

Lemma 2.2. *For any $h > 0$ the operator $K(h)$ has the same eigenvalues* (2.3) *as the operator K. The normalized eigenfunctions of $K(h)$ are*

$$\psi_m^{(h)}(x) = h^{-n/4}\psi_m(h^{-1/2}x), \quad m = 1, 2, \dots, \tag{2.5}$$

where ψ_m are eigenfunctions of K from (2.3′).

Proof: Straightforward calculation. □

2. We shall use some elements of a localization technique from [C-F-K-S] after adding some necessary modifications.

Let us fix a function $J \in C_0^\infty(\mathbf{R}^n)$ such that $0 \le J \le 1$, $J(x) = 1$ if $|x| \le 1$, $J(x) = 0$ if $|x| \ge 2$ and $\tilde{J} = (1 - J^2)^{1/2} \in C^\infty(\mathbf{R}^n)$. Let us fix a number κ, $0 < \kappa < 1/2$, which we shall choose later. For any $h > 0$ define $J^{(h)}(x) = J(h^{-\kappa}x)$. This will be our standard cut-off function.

We shall always suppose that H is an operator of the form (1.2) satisfying the condition (C), and we shall use the notations from Sect.1.

For any $j \in \{1, \dots, N\}$, $\gamma \in \Gamma$ we shall use the canonical coordinates near the point $\gamma\bar{x}_j$ which are translated by the action of γ from an appropriate neighborhood of $\bar{x}_j$. In this way all our coordinate considerations will be Γ-invariant.

For any $j \in \{1, \dots, N\}$, $\gamma \in \Gamma$ and any $m = 1, 2, \dots$ denote by $\psi_{m,j}$ the mth normalized eigenfunction of the model operator K_j and by $\lambda_{m,j}$ the corresponding eigenvalue (they may be repeated according to their multiplicity). We shall also consider the corresponding dilated operator $K_j(h)$ (see (1.10′)) and its normalized eigenfunctions $\psi_{m,j}^{(h)}$ formed like in (2.4) and (2.5).

Using canonical coordinates and a canonical trivialization of E near $\bar{x}_j$ and translating them by γ to a neighborhood of $\gamma\bar{x}_j$, define a truncated eigenfunction $\phi_{m,j,\gamma}$ on M which is supported in a small neighborhood of $\gamma\bar{x}_j$ and given there by the formula

$$\phi_{m,j,\gamma}(x) = J^{(h)}(x)\psi_{m,j}^{(h)}(x). \tag{2.6}$$

By definition this section is supported in a small (of the size $2h^\kappa$) neighborhood of $\gamma\bar{x}_j$.

Lemma 2.3. *For any $M \in \{1, 2, \dots\}$ and all $m, m' \in \{1, \dots, M\}$, $j \in \{1, \dots, N\}$ and $\gamma \in \Gamma$*

$$(\phi_{m,j,\gamma},\ \phi_{m',j',\gamma'}) = \delta_{\gamma,\gamma'}\delta_{j,j'}(\delta_{m,m'} + O(h^\kappa)); \tag{2.7}$$

and

$$(H\phi_{m,j,\gamma},\ \phi_{m',j',\gamma'}) = \delta_{\gamma,\gamma'}\delta_{j,j'}(\lambda_{m,j}\delta_{m,m'} + O(h^{3\kappa-1})) \tag{2.8}$$

uniformly with respect to γ. Here all δ mean the Kronecker symbol, and the inner products are taken in $L^2(M, E)$.

If the operator H is flat near all points $\bar{x}_j$ then instead of (2.7) *and* (2.8) *the following more precise estimates are true:*

$$(\phi_{m,j,\gamma},\ \phi_{m',j',\gamma'}) = \delta_{\gamma,\gamma'}\delta_{j,j'}(\delta_{m,m'} + O(C\exp(-C^{-1}h^{2\kappa-1})); \tag{2.7′}$$

and

$$(2.8') \qquad (H\phi_{m,j,\gamma},\ \phi_{m',j',\gamma'}) = \delta_{\gamma,\gamma'}\delta_{j,j'}(\lambda_{m,j}\delta_{m,m'} + O(C\exp(-C^{-1}h^{2\kappa-1}))\ ,$$

where $C > 0$.

Proof. (a) Obviously the inner products in (2.7) and (2.8) vanish for small h if $\gamma \neq \gamma'$ or $j \neq j'$. Therefore we shall consider the case $\gamma = \gamma',\ j = j'$ only. In this case everything is reduced to the considerations in a small neighborhood of $\bar{x}_j$. There we shall use the canonical coordinates and the canonical trivialization of E. We shall also fix γ and j and skip all subscripts γ and j for simplicity of notations. In particular we shall write ϕ_m, $\psi_m^{(h)}$ and λ_m instead of $\phi_{m,j,\gamma}$, $\psi_{m,j}^{(h)}$ and $\lambda_{m,j}$ respectively.

(b) Let us start by proving (2.7) and (2.7′) (with $j = j'$ and $\gamma = \gamma'$).

Our first concern will be to replace the given volume element $\sqrt{g}dx$ by the Lebesgue measure dx in the given local coordinates. We have

$$(\phi_m, \phi_{m'}) = \int_{|x|\leq 2h^\kappa} (\phi_m(x), \phi_{m'}(x))\sqrt{g}dx = \int_{|x|\leq 2h^\kappa} (\phi_m(x), \phi_{m'}(x))dx + R_1,$$

where

$$R_1 = \int_{|x|\leq 2h^\kappa} (\phi_m(x), \phi_{m'}(x))(\sqrt{g} - 1)dx.$$

Since $\sqrt{g} - 1 = O(|x|) = O(h^\kappa)$ on the support of ϕ_m, we have

$$|R_1| \leq Ch^\kappa \int_{\mathbf{R}^n} |(\psi_m^{(h)}(x), \psi_{m'}^{(h)}(x))|dx \leq Ch^\kappa,$$

which shows that it is sufficient to prove (2.7) for the inner product defined with the use of dx instead of $\sqrt{g}dx$. This new inner product (which depends on the choice of local coordinates) will be denoted $(\cdot, \cdot)_0$.

If H is flat near the points $\bar{x}_j$ then $R_1 = 0$.

(c) As the next step let us replace ϕ_m by $\psi_m^{(h)}$ and estimate the appearing remainder.

We obviously have $\phi_m = \psi_m + \chi_m$ where χ_m vanishes if $|x| \leq h^\kappa$ and $|\chi_m(x)| \leq |\psi_m^{(h)}(x)|$ for all x. Therefore we have

$$(\phi_m, \phi_{m'})_0 = (\psi_m^{(h)}, \psi_{m'}^{(h)})_0 + R = \delta_{m,m'} + R,$$

where

$$|R| \leq 3\int_{|x|\geq h^\kappa} |\psi_m^{(h)}(x)\psi_{m'}^{(h)}(x)|dx = 3\int_{|y|\geq h^{\kappa-1/2}} |\psi_m(y)\psi_{m'}(y)|dy.$$

Since $\kappa < 1/2$, the last integral is $O(C\exp(-C^{-1}h^{2\kappa-1})$ due to (2.2). This proves (2.7) and (2.7′).

(d) Now we shall start proving (2.8) and (2.8′).

Let us write first $H = -hA^{(2)} + H_1$ where H_1 is a first order operator and $A^{(2)}$ has the form (1.6). Using integration by parts, let us write the part of the left-hand side of (2.8) which includes $A^{(2)}$, in the form

$$I_2 = (-hA^{(2)}\phi_m, \phi_{m'}) = h\int_{|x|\leq 2h^\kappa} \sum_{1\leq r,s\leq n} (A_{rs}(x)\frac{\partial\phi_m(x)}{\partial x^s}, \frac{\partial\phi_{m'}(x)}{\partial x^r})\sqrt{g}dx. \tag{2.9}$$

Replacing here $\sqrt{g}$ by 1 we only add $O(hh^{-1/2}h^{-1/2}h^\kappa) = O(h^\kappa)$ which will not interfere with the desired estimates (2.8) and (2.8$'$) because $3\kappa - 1 < \kappa$ and $\sqrt{g} = 1$ near $\bar{x}_j$ if H is flat near $\bar{x}_j$. Similarly we have

$$h((A^{(1)} + A^{(0)})\phi_m, \phi_{m'}) = h((A^{(1)} + A^{(0)})\phi_m, \phi_{m'})_0 + O(h^{\kappa+1/2});$$

$$(B\phi_m, \phi_{m'}) = (B\phi_m, \phi_{m'})_0 + O(h^\kappa);$$
$$(h^{-1}V\phi_m, \phi_{m'}) = (h^{-1}V\phi_m, \phi_{m'})_0 + O(h^{3\kappa-1}),$$

where all remainders vanish if H is flat near $\bar{x}_j$. Therefore we can omit $\sqrt{g}$ in all integrals entering to the inner product (after integrating by parts in the terms initiated by $-hA^{(2)}$).

(e) Let us consider the integral which is obtained if we replace $\sqrt{g}$ by 1 and $A_{rs}(x)$ by $A_{rs}(0)$ in the right hand side of (2.9). This integral will be equal to

$$I_2^0 = (-hA^{(2)}_{flat}\phi_m, \phi_{m'})_0,$$

which is a part of the expression $(K(h)\phi_m, \phi_{m'})_0$ where $K(h) = -hA^{(2)}_{flat} + \bar{B} + h^{-1}V^{(2)}(x)$ is the model operator at the point $\bar{x}_j$ where j was fixed at the beginning of the proof. Then arguments, similar to the ones used in (b), show that $I_2 - I_2^0 = O(h^\kappa)$.

Similarly replacement of B by $\bar{B}$ leads to a remainder term which is $O(h^\kappa)$. Therefore this remainder term can be estimated as $O(h^{3\kappa-1})$ too. Hence it does not interfere with the proof of (2.8) either.

Finally, replacing $h^{-1}V$ by its quadratic part $h^{-1}V^{(2)}$, we add a remainder $O(h^{3\kappa-1})$.

Summarizing these estimates, we obtain

$$(H\phi_m, \phi_{m'}) = (K(h)\phi_m, \phi_{m'})_0 + O(h^{3\kappa-1}). \tag{2.10}$$

Here again the remainder vanishes if H is flat near $\bar{x}_j$.

(f) Now let us replace ϕ_m and $\phi_{m'}$ by $\psi_m^{(h)}$ and $\psi_{m'}^{(h)}$ in the right hand side of (2.10). Arguing as in (c), we shall come to the conclusion that this replacement will only add a remainder $O(C\exp(-C^{-1}h^{2\kappa-1})$. Since $(K(h)\psi_m^{(h)}, \psi_{m'}^{(h)}) = \lambda_m\delta_{m,m'}$, we conclude that the right hand side of (2.10) equals $\lambda_m\delta_{m,m'} + O(h^{3\kappa-1})$ where he remainder can be replaced by $O(C\exp(-C^{-1}h^{2\kappa-1})$ if H is flat near $\bar{x}_j$. This immediately leads to (2.8) and (2.8$'$). □

3. Now we are going to use the variational principle for the von Neumann spectrum distribution function $N_\Gamma(\lambda; H)$. Let us recall its formulation in the context of Hilbert

Γ-modules. For an unbounded operator H we shall denote by $\mathrm{Dom}(H)$ its domain. If H is a linear (unbounded) operator in a Hilbert Γ-module $\mathcal{H}$, then we shall say that H commutes with the action of Γ if $L_\gamma H L_\gamma^{-1} = H$ (this equality includes the coincidence of the domains) for every $\gamma \in \Gamma$. Here L_γ is the operator which determines the action of γ in $\mathcal{H}$. For the case when H is self-adjoint we might say equivalently that every spectral projection of H commutes with all operators L_γ.

Lemma 2.4. (Variational Principle) *Suppose that $\mathcal{H}$ is a Hilbert Γ-module and H is a self-adjoint operator in $\mathcal{H}$ commuting with the action of Γ. Then for every $\lambda \in \mathbf{R}$*

$$N_\Gamma(\lambda; H) = \sup\{\dim_\Gamma L |\ L \subset \mathrm{Dom}(H);\ (Hf, f) \le \lambda(f, f)\ \text{ for all } f \in L\}. \tag{2.11}$$

It is understood here that L should be a Hilbert Γ-submodule in $\mathcal{H}$ (i.e. a closed Γ-invariant subspace in $\mathcal{H}$).

Supposing that H is positive, we can also replace here (Hf, f) by the quadratic form of the operator H and $\mathrm{Dom}(H)$ by the domain of the quadratic form i.e. by $\mathrm{Dom}(H^{1/2})$.

This Lemma is well known (for the proof of a similar statement see e.g. [S1]).

Now let us return to our analytic context and consider the closed subspace $\Phi_\lambda \subset L^2(M, E)$ spanned by all truncated eigenfunctions $\phi_{m,j,\gamma}$ such that $\lambda_{m,j} \le \lambda$. It is obviously Γ-invariant.

Lemma 2.5. *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$*

$$\dim_\Gamma \Phi_\lambda = N(\lambda;\, K)$$

i.e. for small h the *Γ-dimension of Φ_λ coincides with the number of the pairs m, j such that $\lambda_{m,j} \le \lambda$.*

Proof. Consider the (finite-dimensional) linear space Φ_λ^F spanned by the truncated eigenfunctions $\phi_{m,j,e}$ with $\lambda_{m,j} \le \lambda$. Then we obviously have an isomorphism of Hilbert Γ-modules $\Phi_\lambda = \Phi_\lambda^F \bigotimes L^2\Gamma$ (with the trivial action of Γ on the first factor in the right hand side). Therefore $\dim_\Gamma \Phi_\lambda = \dim_{\mathbf{C}} \Phi_\lambda^F$. It remains to prove that for small values of h we have $\dim_{\mathbf{C}} \Phi_\lambda^F = N(\lambda;\, K)$ which amounts to the fact that all the truncated eigenfunctions $\phi_{m,j,e}$ with $\lambda_{m,j} \le \lambda$ are linearly independent. This is obvious due to (2.7). □

Let us check that the subspace Φ_λ can be used in the variational principle.

Lemma 2.6. *Let us fix an arbitrary $R > 0$. For any $\lambda \in [-R, R]$ we have $\Phi_\lambda \subset \mathrm{Dom}(H)$ and there exist $C > 0$ and $h_0 > 0$ such that for any $h \in (0, h_0)$*

$$(Hf, f) \le (\lambda + Ch^{3\kappa-1})(f, f), \quad f \in \Phi_\lambda, \tag{2.12}$$

uniformly with respect to $\lambda \in [-R, R]$. If H is flat near all points $\bar{x}_j$ then (2.12) *can be replaced by*

$$(Hf, f) \le (\lambda + C\exp(-C^{-1}h^{2\kappa-1}))(f, f), \quad f \in \Phi_\lambda, \tag{2.12$'$}$$

Proof. Let us assume first that f is a finite linear combination of the sections $\phi_{m,j,\gamma}$ with $\lambda_{m,j} \le \lambda$, namely,

$$f = \sum c_{m,j,\gamma}\phi_{m,j,\gamma},$$

where $c_{m,j,\lambda}$ are complex constants, such that they all vanish when γ is outside of a finite set (depending on f). Then we have due to (2.7)

$$(f,f) = (\sum c_{m,j,\gamma}\phi_{m,j,\gamma}, \sum c_{m',j',\gamma'}\phi_{m',j',\gamma'}) = \sum c_{m,j,\gamma}\overline{c_{m',j,\gamma}}(\phi_{m,j,\gamma}, \phi_{m',j,\gamma})$$

$$= \sum |c_{m,j,\gamma}|^2(\phi_{m,j,\gamma}, \phi_{m,j,\gamma}) + \sum_{m\neq m'} c_{m,j,\gamma}\overline{c_{m',j,\gamma}}(\phi_{m,j,\gamma}, \phi_{m',j,\gamma})$$

$$= (1+O(h^\kappa))\sum |c_{m,j,\gamma}|^2 + O(h^\kappa)\sum_{m\neq m'} |c_{m,j,\gamma}c_{m',j,\gamma}| = (1+O(h^\kappa))\sum |c_{m,j,\gamma}|^2.$$

Similarly, using (2.8) we obtain
$(Hf,f) =$

$$(1+O(h^{3\kappa-1}))\sum \lambda_{m,j}|c_{m,j,\gamma}|^2 + O(h^{3\kappa-1})\sum |c_{m,j,\gamma}|^2 \le (\lambda + O(h^{3\kappa-1}))\sum |c_{m,j,\gamma}|^2.$$

Therefore

$$(2.13) \qquad (Hf,f) \le (\lambda + O(h^{3\kappa-1}))(1+O(h^\kappa))(f,f) \le (\lambda + O(h^{3\kappa-1})(f,f).$$

Similar arguments show that $(Hf,Hf) \le (\lambda^2 + O(h^{3\kappa-1}))(f,f)$. Using the closedness of the operator H and taking a limit from the finite sums we easily conclude that $\Phi_\lambda \subset \mathrm{Dom}(H)$. The same limit transition shows that (2.12) holds for all $f \in \Phi_\lambda$.

Using in this argument (2.7′) and (2.8′) instead of (2.7) and (2.8) we obtain (2.12′). □

Using the space Φ_λ in the variational principle (Lemma 2.4), we immediately obtain

Proposition 2.7. *For any $\kappa \in (0,1/2)$ there exist $C > 0$ and $h_0 > 0$ such that for any $\lambda \in [-R,R]$ and $h \in (0,h_0)$*

$$(2.14) \qquad N_\Gamma(\lambda + Ch^\kappa; H) \ge N(\lambda; K)\ .$$

If H is flat near all points $\bar{x}_j$ then instead of (2.14) *we can write*

$$(2.14') \qquad N_\Gamma(\lambda + C\exp(-C^{-1}h^{-1+\varepsilon}); H) \ge N(\lambda; K)$$

for any $\varepsilon > 0$ (with $C > 0$ depending also on ε).

Proof. Lemmas 2.4 and 2.6 yield (2.14) with $h^{3\kappa-1}$ instead of h^κ. But $3\kappa - 1 \to 1/2$ as $\kappa \to 1/2$, so $3\kappa - 1$ can be done as close to $1/2$ as we like by an appropriate choice of κ.

Similarly if H is flat near all points $\bar{x}_j$ we should choose $\kappa > 0$ sufficiently small to arrive to (2.14′). □

Corollary 2.8. *For any $\kappa \in (0, 1/2)$ there exist $C > 0$ and $h_0 > 0$ such that for any $\lambda \in [-R, R]$ and $h \in (0, h_0)$*

$$N_\Gamma(\lambda; H) \geq N(\lambda - Ch^\kappa; K). \tag{2.15}$$

If H is flat near all points $\bar{x}_j$ then instead of (2.15) *we can write*

$$N_\Gamma(\lambda; H) \geq N(\lambda - C\exp(-C^{-1}h^{-1+\varepsilon}); K) \tag{2.15$'$}$$

for any $\varepsilon > 0$.

Proof. Replace λ by $\lambda - Ch^\kappa$ (resp. by $\lambda - C\exp(-C^{-1}h^{-1+\varepsilon})$) in (2.14) (resp. (2.14′)). □

3. Estimate from above.

1. We shall start by establishing a version of so called IMS localization formula (see [C-F-K-S] and [Si2]; the abbreviation IMS is composed from first letters of names of R.Ismagilov, J.Morgan, I.Sigal and B.Simon).

We shall use the same cut-off function $J^{(h)}$ as in Sect.2 (constructed with a choice of $\kappa \in (0, 1/2)$ to be specified later). We shall also use the same Γ-invariant set of canonical coordinates and trivializations of E near the points $\gamma\bar{x}_j,\ \gamma \in \Gamma,\ j = 1, \dots, N$. Define a cut-off function $J_{j,\gamma}$ which is supported near $\gamma\bar{x}_j$ and equals $J^{(h)}$ in the corresponding canonical coordinates. Denote also $J_0(x) = \sqrt{1 - \sum_{j,\gamma} J_{j,\gamma}^2(x)}$. Then $J_0 \in C^\infty(M)$ and

$$J_0^2(x) + \sum_{j,\gamma} J_{j,\gamma}^2(x) = 1. \tag{3.1}$$

We shall identify the functions $J_0,\ J_{j,\gamma}$ with the corresponding multiplication operators.

Lemma 3.1. *The following operator identity is true:*

$$H = J_0 H J_0 + \sum_{j,\gamma} J_{j,\gamma} H J_{j,\gamma} - ha^{(2)}(x, dJ_0(x)) - h\sum_{j,\gamma} a^{(2)}(x, dJ_{j,\gamma}(x))\ , \tag{3.2}$$

where $a^{(2)}$ is the principal symbol of $-A$ (see (1.7)*) and the terms with $a^{(2)}$ are endomorphisms of the bundle E.*

Proof. Using (3.1) we can write

$$H = J_0^2 H + \sum_{j,\gamma} J_{j,\gamma}^2 H = J_0 H J_0 + \sum_{j,\gamma} J_{j,\gamma} H J_{j,\gamma} + J_0[J_0, H] + \sum_{j,\gamma} J_{j,\gamma}[J_{j,\gamma}, H].$$

Similarly,

$$H = HJ_0^2 + \sum_{j,\gamma} HJ_{j,\gamma}^2 = J_0 H J_0 + \sum_{j,\gamma} J_{j,\gamma} H J_{j,\gamma} - [J_0, H]J_0 - \sum_{j,\gamma}[J_{j,\gamma}, H]J_{j,\gamma}.$$

Summing these identities and dividing by 2 we come to the identity

$$H = J_0 H J_0 + \sum_{j,\gamma} J_{j,\gamma} H J_{j,\gamma} + \frac{1}{2}[J_0, [J_0, H]] + \frac{1}{2}\sum_{j,\gamma}[J_{j,\gamma}, [J_{j,\gamma}, H]] \,.$$

It remains to notice that $[J,[J,H]]$ coincides with the bundle endomorphism $-2ha^{(2)}(x, dJ(x))$ for any $J \in C^\infty(M)$. □

Corollary 3.2. *There exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$*

$$H \geq J_0 H J_0 + \sum_{j,\gamma} J_{j,\gamma} H J_{j,\gamma} - Ch^{1-2\kappa} I. \tag{3.3}$$

Proof. The first derivatives of the cut-off functions that we use in (3.2) can be estimated as $O(h^{-\kappa})$. Hence the terms with $a^{(2)}$ have the operator norm $O(h^{1-2\kappa})$. This immediately implies (3.3). □

2. Let us estimate from below terms in the right hand side of (3.3). We shall start with the term containing J_0.

Lemma 3.3. *There exist $c > 0$ and $h_0 > 0$ such that for any $h \in (0, h_0)$*

$$J_0 H J_0 \geq ch^{-1+2\kappa} J_0^2 I. \tag{3.4}$$

Proof. Since $-hA$ and B are semibounded from below, it is sufficient to prove that $h^{-1} V J_0^2 \geq ch^{-1+2\kappa} I$, which immediately follows from (1.5) and the definition of J_0. □

In estimating from below of the term localized near $\gamma\bar{x}_j$, we will use the model operator $K_j(h)$ which has the form

$$K_j(h) = -hA^{(2)}_{flat} + \bar{B}_j + h^{-1} V^{(2)}(x), \tag{3.5}$$

where all components are taken in the canonical coordinates and the canonical trivialization near $\bar{x}_j$ (like in (1.10)) and translated to $\gamma\bar{x}_j$ by the action of γ.

Lemma 3.4. *There exist $C > 0$ and $h_0 > 0$ such that for any $h \in (0, h_0)$, $j \in \{1, \dots, N\}$ and $\gamma \in \Gamma$*

$$J_{j,\gamma} H J_{j,\gamma} \geq (1 - Ch^\kappa) J_{j,\gamma} K_{j,\gamma} J_{j,\gamma} - Ch^{3\kappa-1} J_{j,\gamma}^2 \tag{3.6}$$

or

$$(HJ_{j,\gamma}u, J_{j,\gamma}u) \geq (1 - Ch^\kappa)(K_{j,\gamma} J_{j,\gamma}u, J_{j,\gamma}u) - Ch^{3\kappa-1}(J_{j,\gamma}u, J_{j,\gamma}u) \tag{3.6$'$}$$

for any $u \in \mathrm{Dom}(H)$. The inner product $(\cdot, \cdot)$ can be replaced by the flat inner product $(\cdot, \cdot)_0$ taken in the corresponding canonical local coordinates.

If H is flat near $\bar{x}_j$ then

$$J_{j,\gamma} H J_{j,\gamma} = J_{j,\gamma} K_{j,\gamma} J_{j,\gamma} \ .$$

Proof. Fixing j and γ in the proof of this Lemma we shall omit the subscripts. We shall make all considerations in the canonical coordinates and the canonical trivialization near the point $\bar{x} = \gamma \bar{x}_j$.

We want to estimate from below the quadratic form $(JHJu, u)$ where $u \in \mathrm{Dom}(H)$, in terms of $(JKJu, u)$. (Here $K = K_j$ and $J = J_{j,\gamma}$.)

We shall use notations from the proof of Lemma 2.3 and argue similarly. Let us start with the term

$$(-hJA^{(2)}Ju, u) = h \int_{|x| \leq 2h^\kappa} \sum_{1 \leq r,s \leq n} (A_{rs}(x) \frac{\partial (Ju)}{\partial x^s}, \frac{\partial \overline{(Ju)}}{\partial x^r}) \sqrt{g} dx.$$

Replacing $\sqrt{g}$ by 1 and $A_{rs}(x)$ by $A_{rs}(0)$ we add terms of similar form but with additional factor $O(h^\kappa)$. It follows that there exist $C > 0$ and $h_0 > 0$ such that for any $h \in (0, h_0)$

$$(-hJA^{(2)}Ju, u) \geq (1 - Ch^\kappa)(-hJA^{(2)}_{flat}Ju, u)_0. \tag{3.7}$$

Similarly in all other terms replacing $\sqrt{g}$ by 1 leads to admissible remainders, so we can actually omit the subscript 0 at the inner product (the more convenient inner product is usually clear from the context).

Let us estimate the term $(-hJA^{(1)}Ju, u)$. Denote by $D(Ju)$ the $k \times n$ matrix of all first partial derivatives of all components of the vector Ju. Let $\|D(Ju)\|_0$ be the L^2 norm (in local coordinates) of Du considered as a vector function. Similarly denote by $\|Ju\|_0$ the L^2-norm of Ju. Then we obviously have for every $\varepsilon > 0$

$$|(-hJA^{(1)}Ju, u)| \leq Ch\|D(Ju)\|_0 \|Ju\|_0 \leq Ch\varepsilon \|D(Ju)\|_0^2 + Ch\varepsilon^{-1} \|Ju\|_0^2.$$

Taking $\varepsilon = h^\kappa$ with we obtain

$$|(-hJA^{(1)}Ju, u)| \leq Ch^\kappa (-hJA^{(2)}_{flat}Ju, u)_0 + Ch^{1-\kappa}(Ju, Ju), \quad h \in (0, h_0).$$

Also obviously

$$|(-hJA^{(0)}Ju, u)| \leq h(Ju, Ju).$$

Therefore (3.7) implies for small h

$$(-hJAJu, u) \geq (1 - Ch^\kappa)(-hJA^{(2)}_{flat}Ju, u)_0 - Ch^{1-\kappa}(J^2u, u). \tag{3.8}$$

Replacing $B(x)$ by $\bar{B}$ in the quadratic form $(JBJu, u)$ contributes a term which can be estimated by $Ch^\kappa (J^2 u, u)$.

Finally, we have

$$h^{-1}(JVJu, u) \geq h^{-1}(JV^{(2)}Ju, u)_0 - Ch^{3\kappa-1}(Ju, Ju).$$

Gathering together all these estimates, we obtain (3.6). □

2. Now we shall discuss some preliminaries about rank of morphisms of Hilbert Γ-modules. We refer the reader to [C-G1, C, D, L-L, S6, T] for more details.

If L_1 and L_2 are Hilbert Γ-modules, then a bounded linear operator $A : L_1 \to L_2$ is called a *morphism* of Hilbert Γ-modules if it commutes with the action of Γ. We shall need the following simple facts:

(i) *If* $\operatorname{Ker} A = 0$ *then* $\dim_\Gamma L_1 \leq \dim_\Gamma L_2$.

(ii) *If* $\operatorname{Im} A$ *is dense in* L_2 *then* $\dim_\Gamma L_1 \geq \dim_\Gamma L_2$.

Definition 3.5. $\operatorname{rank}_\Gamma A = \dim_\Gamma \overline{\operatorname{Im} A}$, *where the bar means the norm closure.*

Lemma 3.6. (a) *If* $A, B : L_1 \to L_2$ *are morphisms of Hilbert* Γ*-modules, then*

$$\operatorname{rank}_\Gamma(A+B) \leq \operatorname{rank}_\Gamma A + \operatorname{rank}_\Gamma B.$$

(b) *If* $D : L_1 \to L_2$, $A : L_2 \to L_3$ *and* $C : L_3 \to L_4$ *are morphisms of Hilbert* Γ*-modules, then*

$$\operatorname{rank}_\Gamma(CAD) \leq \operatorname{rank}_\Gamma A.$$

Proof. (a) The morphism

$$\overline{\operatorname{Im} A} \oplus \overline{\operatorname{Im} B} \to \overline{\operatorname{Im}(A+B)}, \quad \{u, v\} \mapsto u + v,$$

has a dense image, so it is sufficient to apply (i) above.

(b) It is sufficient to prove that

$$\operatorname{rank}_\Gamma(CA) \leq \operatorname{rank}_\Gamma A \quad \text{and} \quad \operatorname{rank}_\Gamma(AD) \leq \operatorname{rank}_\Gamma A.$$

Note that the morphism $C : \overline{\operatorname{Im} A} \to \overline{\operatorname{Im}(CA)}$ has a dense image which proves the first inequality due to (ii) above. To prove the second one, note that $\operatorname{Im}(AD) \subset \operatorname{Im} A$, therefore $\overline{\operatorname{Im}(AD)} \subset \overline{\operatorname{Im} A}$, so it is sufficient to apply the monotonicity of the Γ-dimension. □

Lemma 3.7. *Assume that* H *is a self-adjoint operator in a Hilbert* Γ*-module* $\mathcal{H}$ *and* H *commutes with the action of* Γ. *Suppose that there exists an endomorphism* T *of the Hilbert* Γ*-module* L *such that*

$$((H+T)u, u) \geq \mu(u, u), \ u \in \operatorname{Dom}(H) \quad \text{and} \quad \operatorname{rank}_\Gamma T \leq k. \tag{3.9}$$

Then

$$N_\Gamma(\mu - \varepsilon;\, H) \leq k \quad \text{for any} \quad \varepsilon > 0.$$

Proof. We shall use the variational principle (Lemma 2.4). Assume that L is a closed Γ-invariant subspace in $\mathcal{H}$ such that $L \subset \mathrm{Dom}(H)$ and

$$(Hu, u) \leq (\mu - \varepsilon)(u, u), \quad u \in L.$$

We have to prove that then $\dim_\Gamma L \leq k$. Arguing by contradiction, assume that $\dim_\Gamma L > k$. Consider then restriction T_L of T to L as a morphism $T_L : L \to \overline{\mathrm{Im}\, T}$. Since $\dim_\Gamma \overline{\mathrm{Im}\, T} = \mathrm{rank}_\Gamma T \leq k$, the map T_L can not be injective. Hence there exists $u \in L - \{0\}$ such that $Tu = 0$. Therefore

$$((H+T)u, u) = (Hu, u) \leq (\mu - \varepsilon)(u, u),$$

which contradicts to (3.9). □

3. We shall construct an operator T which will allow us to apply Lemma 3.7 in our analytic context. To do this consider first for every $j \in \{1, \ldots, N\}$ the model operator $K_j(h)$ and take for every $\lambda \in \mathbf{R}$ its spectral projection $E_\lambda^{(j)}$. This is a finite rank operator in $L^2(\mathbf{R}^n) \otimes \mathbf{C}^k$ and $\mathrm{rank}\, E_\lambda^{(j)} = N(\lambda; K_j(h)) = N(\lambda; K_j)$ (the number of eigenvalues of K_j which are $\leq \lambda$). If a constant $M > 0$ is sufficiently large then $K_j(h) + ME_\lambda^{(j)} \geq \lambda I$. (It is sufficient to take $M \geq \lambda - \inf \mathrm{spec}(K_j)$.)

In the canonical local coordinates near $\gamma \bar{x}_j$ we can consider the operator $T_{j,\gamma} = J_{j,\gamma} M E_\lambda^{(j)} J_{j,\gamma}$ and then

$$J_{j,\gamma}\left(K_j(h) + ME_\lambda^{(j)}\right) J_{j,\gamma} = J_{j,\gamma} K_j(h) J_{j,\gamma} + T_{j,\gamma} \geq \lambda J_{j,\gamma}^2 I. \tag{3.10}$$

Now take

$$T = \sum_{j,\gamma} T_{j,\gamma}.$$

Lemma 3.8. *For any $\kappa \in (0, 1/2)$ and any $R > 0$ there exist $C > 0$ and $h_0 > 0$ such that for any $\lambda \in [-R, R]$ and any $h \in (0, h_0)$*

$$H + T \geq (\lambda - Ch^s) I, \tag{3.11}$$

where $s = \min\{3\kappa - 1, 1 - 2\kappa\}$. If H is flat near all the points $\bar{x}_j$ then instead of (3.11) we have

$$H + T \geq (\lambda - Ch^{1-\kappa}) I\ . \tag{3.11$'$}$$

Proof. Using Corollary 3.2, Lemmas 3.3 and 3.4, (3.10) and (3.1), we get

$$H + T \geq (1 - Ch^\kappa)\lambda I - Ch^{1-2\kappa} I - Ch^{3\kappa-1} I$$

in the general case and (3.11′) if H is flat near all the points $\bar{x}_j$. It remains to notice that $3\kappa - 1 < \kappa$. □

Corollary 3.9. *For any $R > 0$ there exist $C > 0$ and $h_0 > 0$ such that for any $\lambda \in [-R, R]$ and $h \in (0, h_0)$*

$$H + T \geq (\lambda - Ch^{1/5})I. \tag{3.12}$$

Proof. We should apply Lemma 3.8 with the best possible (i.e. biggest) value of s which is

$$s = \max_{\kappa} \min\{3\kappa - 1, 1 - 2\kappa\} = 1/5.$$

(The maximum is attained when $3\kappa - 1 = 1 - 2\kappa$ i.e. $\kappa = 2/5$.) □

Lemma 3.10. *For the operator T constructed above*

$$\text{rank}_\Gamma T \leq N(\lambda; K). \tag{3.13}$$

Proof. First consider $T_j = \sum_\gamma T_{j,\gamma}$. Then $\text{rank}_\Gamma T_j = \text{rank}\, R_{j,\gamma}$ because T_j is just a tensor product of $T_{j,\gamma}$ and the identity operator in $L^2\Gamma$. Now using Lemma 3.6 we obtain

$$\text{rank}_\Gamma T \leq \sum_{j=1}^{N} \text{rank}_\Gamma T_j \leq \sum_{j=1}^{N} \text{rank}\, E_\lambda^{(j)} = N(\lambda; K). \quad \square$$

Now we can prove the main estimate from above:

Proposition 3.11. *For any $R > 0$ there exist $C > 0$ and $h_0 > 0$ such that for any $\lambda \in [-R, R]$ and any $h \in (0, h_0)$*

$$N_\Gamma(\lambda; H) \leq N(\lambda + Ch^{1/5}; K) \tag{3.14}$$

and

$$N_\Gamma(\lambda; H) \leq N(\lambda + Ch^{1-\kappa}; K) \tag{3.14$'$}$$

if H is flat near all the points $\bar{x}_j$.

Proof. Using Lemma 3.8, Corollary 3.9 and Lemma 3.10 we can apply Lemma 3.7 with $\mu = \lambda - Ch^{1/5}$ and $k = N(\lambda; K)$. We obtain then

$$N_\Gamma(\lambda - Ch^{1/5} - \varepsilon; H) \leq N(\lambda; K) \quad \text{for every } \varepsilon > 0.$$

Replacing λ by $\lambda + Ch^{1/5} + \varepsilon$ we obtain

$$N_\Gamma(\lambda; H) \leq N(\lambda + Ch^{1/5} + \varepsilon; K) \quad \text{for every } \varepsilon > 0.$$

But then (3.14) immediately follows if we take limit as $\varepsilon \to 0$ using the fact that both functions $N_\Gamma(\cdot; H)$ and $N(\cdot; K)$ are by definition right continuous.

Similarly if H is flat near all the points $\bar{x}_j$, then we can apply Lemma 3.7 with $\mu = \lambda - Ch^{1-\kappa}$ and $k = N(\lambda; K)$ which leads to (3.14′). □

Proof of Theorem 1.5 and estimate (1.20). Both immediately follow from the estimates for $N_\Gamma(\lambda; H)$ given in Corollary 2.8 and Proposition 3.11. □

4. Morse inequalities.

1. Our proof of Theorems 1.10 and 1.11 will be based on the ideas of E.Witten [W1] (see also [Si2], [C-F-K-S], [H2], [H-S2], [H]). We shall use the Witten deformation of the Laplacian on differential forms on the covering manifold, and apply the semiclassical asymptotics near 0 from Theorem 1.5 (or rather Corollaries 1.8 and 1.9) to the deformed Laplacian.

Let us introduce necessary notations.

Let M be a riemannian manifold with a discrete group of isometries Γ which acts free on M so that M/Γ is compact. Let $\Delta_p = -(dd^* + d^*d)$ be the Laplacian on $\Lambda^p(M)$. Let f be a Γ-invariant Morse function on M. Denote

$$d_h = e^{-f/h} d e^{f/h}, \quad d_h^* = e^{f/h} d^* e^{-f/h}; \tag{4.1}$$

so d_h and d_h^* are formally adjoint operators in $\oplus_p L^2\Lambda^p(M)$ where $L^2\Lambda^p(M) = L^2(M, \Lambda^p T^*(M))$. Note that $d_h^2 = 0$ and $(d_h^*)^2 = 0$.

The Witten deformation (of $-\Delta$) will be then the following hamiltonian depending on a (small) parameter $h > 0$:

$$H_p = h(d_h + d_h^*)^2 = h(d_h d_h^* + d_h^* d_h). \tag{4.2}$$

It is a non-negative self-adjoint operator. Moreover $H_p = -hA + B + h^{-1}V$ where

$$A = \Delta, \quad B = \mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^* \quad \text{and} \quad V = |df|^2.$$

Here ∇f is the gradient vector field of f, $\mathcal{L}_{\nabla f}$ is the Lie derivative along this field and $\mathcal{L}_{\nabla f}^*$ is the formally adjoint operator. It is easy to check that B is really a zero order operator. The operator V is just a multiplication by a scalar function $V(x) = |df|^2(x)$. We see that H_p really has the form allowing application of Theorem 1.5 (and Corollaries 1.8, 1.9). Denote by K_p the corresponding model operator. The problem of calculating its eigenvalues actually reduces to the diagonalizing of B at the critical points of f. This was done by E.Witten and the part of the result which we need is the following

Lemma 4.1. ([W1], [H-S2]) *The multiplicity of* 0 *as an eigenvalue of* K_p *is equal to* m_p *i.e. to the number of the critical points of the function* f *with the index* p *in a fundamental domain of* Γ.

It follows from Corollary 1.8 that the von Neumann dimension of the spectral subspace of H_p corresponding to the interval $[0, Ch^{1/5}]$ of the spectral line, equals m_p. Using this fact (or Corollary 1.9) we obtain

$$m_p \geq \dim_\Gamma \operatorname{Ker} H_p. \tag{4.3}$$

Lemma 4.2. *For every $p = 0, 1, \ldots, n$ and every $h > 0$*

$$\dim_\Gamma \operatorname{Ker} H_p = \dim_\Gamma \operatorname{Ker} \Delta_p = \bar{b}_p.$$

(*Here H_p and Δ_p are considered as self-adjoint operators in $L^2\Lambda^p(M)$.*)

Proof. For every fixed $h > 0$ we have the Hodge-type decomposition

$$L^2\Lambda^p(M) = \overline{d_h\Lambda_c^{p-1}(M)} + \operatorname{Ker} H_p + \overline{d_h^*\Lambda_c^{p+1}(M)}, \tag{4.4}$$

where $\Lambda_c^k(M)$ is the space of all k-forms with a compact support on M and the bars mean closures in L^2. To prove it we should notice that the first and the third spaces in the right hand side of (4.4) are orthogonal due to the fact that $d_h^2 = 0$, and the orthogonal complement to the direct sum of these two spaces equals

$$\operatorname{Ker} d_h \cap \operatorname{Ker} d_h^* = \operatorname{Ker} H_p. \tag{4.5}$$

Note also that $\operatorname{Ker} d_h$ in $L^2\Lambda^p(M)$ coincides with the orthogonal complement to the third space in the right hand side of (4.4), so it is actually the direct sum of the first two spaces. Therefore we obtain a linear topological isomorphism of Γ-modules

$$\operatorname{Ker} H_p = \operatorname{Ker} d_h|_{L^2\Lambda_p(M)}/\overline{d_h\Lambda_c^{p-1}(M)}. \tag{4.6}$$

The space in the right hand side actually does not depend on h and is Γ-isomorphic to the space of the reduced p-cohomologies which is its particular case when $h = 0$, because the multiplication operator by $e^{f/h}$ maps isomorphically $\operatorname{Ker} d_h$ to $\operatorname{Ker} d$ and $\operatorname{Im} d_h$ to $\operatorname{Im} d$. □

2. We shall need some preparatory remarks about complexes of Hilbert Γ-modules. Let us consider a sequence

$$L:\ 0 \longrightarrow L_0 \xrightarrow{d_0} L_1 \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} L_n \longrightarrow 0. \tag{4.7}$$

It is called a *complex of Hilbert Γ-modules* if all $L_j,\ j = 1, \ldots n$, are Hilbert Γ-modules and $d_j,\ j = 1, \ldots, n-1$, are morphisms of Hilbert Γ-modules and $d_j d_{j-1} = 0$ for all j.

The *reduced cohomologies* of L are defined as the Hilbert Γ-modules

$$\bar{H}^p(L) = \operatorname{Ker} d_p/\overline{\operatorname{Im} d_{p-1}}, \quad p = 0, 1, \ldots, n.$$

(Here by definition d_{-1} and d_n are zero morphisms.) Denote $m_p = \dim_\Gamma L_p$, $\bar{b}_p = \dim_\Gamma \bar{H}^p(L)$.

The following Lemma is well known:

Lemma 4.3. *Suppose that $m_p < \infty$ for all p. Then*

$$\sum_{p=0}^{n} (-1)^p m_p = \sum_{p=0}^{n} (-1)^p \bar{b}_p. \tag{4.8}$$

The proof does not differ from the proof of the corresponding statement for the trivial group Γ (and for the spaces L_j which are finite-dimensional in the usual sense) except almost isomorphisms should be used instead of usual isomorphisms (see [S6] for more details).

Lemma 4.4. *If $m_j < \infty$ for all j then*

$$\sum_{j=0}^{p}(-1)^{p-j}m_j \geq \sum_{j=0}^{p}(-1)^{p-j}\bar{b}_j \tag{4.9}$$

for every $p = 0, \dots n$.

Proof. Let us consider a complex of Hilbert Γ-modules

$$L^{(p)}:\ 0 \longrightarrow L_0 \xrightarrow{d_0} L_1 \xrightarrow{d_1} \dots \xrightarrow{d_{p-1}} L_p \longrightarrow\xrightarrow{d_p} \overline{\operatorname{Im} d_p} \longrightarrow 0.$$

Then $\bar{H}^j(L^{(p)}) = \bar{H}^j(L),\ j = 0, \dots, p$, and $\bar{H}^j(L^{(p)}) = 0$ if $j > p$. Therefore Lemma 4.3 yields in this case

$$\sum_{j=0}^{p}(-1)^{p-j}m_j - \dim_\Gamma \overline{\operatorname{Im} d_p} = \sum_{j=0}^{p}(-1)^{p-j}\bar{b}_j.$$

Now (4.9) immediately follows. □

Proof of Theorem 1.10. Let us consider a complex of the form (4.7) where $L_p = \operatorname{Im} E([0, Ch^{1/5}]; H_p)$ and d_p is the deformed de Rham differential from (4.1) (denoted there by d_h). The differentials map L_p to L_{p+1} because they commute with H_p. (More precisely, $d_pH_p = H_{p+1}d_p$ on L_p). Moreover these differentials are obviously norm bounded (by $Ch^{1/5}$), hence they are morphisms of Hilbert Γ-modules. Therefore Theorem 1.10 follows from Corollary 1.8 and Lemmas 4.1–4.4. □

3. Proof of Theorem 1.11. We shall use notations introduced in Sect. 1.4. Both operators $\Delta^\pm_\omega$ have the form $H = -hA + B + h^{-1}V$ required to apply Theorem 1.5 and Corollaries 1.8 and 1.9. Here $A = \Delta^\pm$ where Δ^+ (Δ^-) is the usual Laplacian of the chosen riemannian metric acting on the forms of even (resp. odd) degree; V is the multiplication operator by the scalar function $V(x) = |\mathbf{v}(x)|^2 = |\omega(x)|^2$; B is a zero order matrix operator which has the form

$$B = \mathcal{L}_\mathbf{v} + \mathcal{L}^*_\mathbf{v} + \lambda_{d\omega} + \lambda^*_{d\omega},$$

where $\mathcal{L}$ means the Lie derivative and $\lambda_{d\omega}$ is the external multiplication by the form $d\omega$. We see that $V(x)$ vanishes precisely at the singular points of the vector field $\mathbf{v}$.

Here again the calculation of the eigenvalues of the model operators $K^\pm$ at a singular point $\bar{x}$ reduces to the calculation of the eigenvalues of the matrix $B(\bar{x})$. This was done by S.Novikov (see Appendix to [N-S1] and also [S7]) with the help of the canonical fermionic Bogolyubov transformation. From this calculation we need only the fact that the multiplicity of 0 as an eigenvalue of $K^\pm$ equals $m_\pm$.

Now we can apply Corollary 1.8 to conclude that

$$\dim_\Gamma \operatorname{Im} E([0, Ch^{1/5}]; \Delta_\omega^\pm) = m^\pm$$

if $h \in (0, h_0)$ and h_0 is sufficiently small. Therefore

$$m_\pm \geq \bar{b}_\pm(h) \quad \text{for all} \quad h \in (0, h_0).$$

The inequalities (1.30) immediately follow. The equality (1.31) follows from the Atiyah L^2 index theorem (see [At]) applied to the elliptic Γ-invariant operator on M

$$d_h + d_h^* : \Lambda^{ev}(M) \longrightarrow \Lambda^{odd}(M)$$

because the index of this operator does not depend on h which enters only into lower order terms. □

Appendix.

In this Appendix we give a sketch of the proof for the exponential estimate (2.2) of the eigenfunctions of the model matrix quadratic operator (2.1). The main idea is a use of exponential weight functions of the form $w_a(x) = \exp(a|x|^2)$. Denote by W_a the multiplication operator by $w_a(\cdot)$ in vector functions on $\mathbf{R}^n$ (more precisely, W_a is the multiplication by $w_a(\cdot)I$ where I is the identity endomorphism of $\mathbf{C}^k$) and let $\psi_a = W_a\psi$. Due to the standard elliptic regularity results it is sufficient to prove that for any eigenfunction ψ of K there exists $a > 0$ such that $\psi_a \in L^2(\mathbf{R}^n) \otimes \mathbf{C}^k$. Denote $K_a = W_a K W_{-a}$. Obviously, K_a is again a differential second order elliptic operator (with the same principal part) and $K\psi = E\psi$ if and only if $K_a\psi_a = E\psi_a$.

We shall take $a \in \mathbf{C}$ sufficiently small and consider K_a as a perturbation of the operator $K = K_0$. The most important information about this perturbation is contained in the following

Proposition A.1. *There exist* $\varepsilon > 0$ *and* $C > 0$ *such that for every* a *with* $|a| < \varepsilon$ *we have* $\operatorname{Dom}(K_a) = \operatorname{Dom}(K)$ *and the following estimate is true*:

$$\|(K_a - K)u\| \leq Ca\|Ku\| + C\|u\|, \quad u \in \operatorname{Dom}(K) . \tag{A.1}$$

Proof. Using the notations from (2.1) we see that

$$K_a = -W_a A(D) W_{-a} + B + V(x) ,$$

because B and $V(x)$ are zero order operators. It is easy to see that

$$K_a - K = -W_a A(D) W_{-a} + A(D)$$

is a finite sum of terms of the following forms:

$$aM, \quad a^2 x^r x^s M, \quad a x^r M \frac{\partial}{\partial x^s} ,$$

where M is a constant matrix. Denote one of these terms by aT, so T is a matrix differential operator of order ≤ 1 whose coefficients are polynomials of degree ≤ 1 by the first derivatives and of degree ≤ 2 when there is no derivatives in the term. It follows from the theory of global elliptic operators ([S4], [H1]), that if a constant μ_0 is chosen so that $\mu_0 < \inf \operatorname{spec} K$ then the operator $T(K-\mu_0)^{-1}$ is bounded in $L^2(\mathbf{R}^n)\otimes \mathbf{C}^k$. This means that T is defined on $\operatorname{Im}(K-\mu_0)^{-1} = \operatorname{Dom}(K)$ and there exists $C>0$ such that

$$\|T(K-\mu_0)^{-1}v\| \leq C\|v\|, \quad v \in L^2(\mathbf{R}^n)\otimes \mathbf{R}^k \ .$$

Denoting $(K-\mu_0)^{-1}v = u$ we come to the estimate

$$\|Tu\| \leq C\|Ku\| + C\mu_0\|u\|, \quad u \in \operatorname{Dom}(K),$$

and the estimate (A.1) immediately follows. □

Let us recall the terminology from [R-S], Ch. XIII.11. An *analytic family of type* (A) is a family $T(\beta)$ of closed operators depending on the complex parameter β which runs over an open set $\Omega \subset \mathbf{C}$ such that the following three conditions are fulfilled:

(i) $T(\beta)$ has non-empty resolvent set for every $\beta \in \Omega$;

(ii) $\operatorname{Dom}(T(\beta)) = D$ does not depend on β;

(iii) for every $\psi \in D$ the function $\beta \mapsto T(\beta)\psi$ is a vector-valued analytic function of $\beta \in \Omega$.

Any analytic family of type (A) is an analytic family in the sense of Kato. This means that the condition (i) is satisfied, for every $\beta_0 \in \Omega$ there exists $\lambda_0 \notin \operatorname{spec}(T(\beta_0))$ such that $\lambda_0 \notin \operatorname{spec}(T(\beta))$ for every β in a neighborhood of β_0 and the resolvent $\beta \mapsto (T(\beta)-\lambda_0)^{-1}$ is an analytic operator-valued function of β in a neighborhood of β_0.

Using this terminology we see that Proposition A.1 immediately imply

Corollary A.2. *The operator-function $a \mapsto K_a$ is a regular analytic perturbation of H of type* (A) (*in particular it is an analytic perturbation in the sense of Kato*).

Proof. See [K] or [R-S], Ch.XIII.11. □

Proposition A.3. *If H is the model self-adjoint matrix quadratic operator of the form* (2.1) *and $\psi \in L^2(\mathbf{R}^n)$ is its eigenfunction i.e. $H\psi = E\psi$ where E is a constant, then there exist $a>0$ and $C>0$ such that*

$$|\psi(x)| \leq C\exp(-a|x|^2) \ . \tag{A.2}$$

Proof. Denote by m the multiplicity of the eigenvalue E for the operator K. Using the regular analytic perturbation theory for self-adjoint operators (see [K] or [R-S], Ch.XIII.11) we see that there exist $\varepsilon>0$ and $\delta>0$ such that for any $a\in\mathbf{C}$ with $|a|<\delta$ the operator K_a has exactly m generalized eigenvalues (multiplicities counted) in the disk $|\tilde{E}-E|<\varepsilon$.

Suppose that $0<a<\delta$ and $\phi \in L^2(\mathbf{R}^n)\otimes \mathbf{C}^k$ is a generalized eigenvector of K_a with the eigenvalue $\tilde{E}$, $|\tilde{E}-E|<\varepsilon$. This means that $\phi \neq 0$ and $(H_a-\tilde{E})^p\phi = 0$ for some positive integer p. Equivalently we can rewrite this as follows:

$$e^{a|x|^2}(K-\tilde{E})^p e^{-a|x|^2}\phi = 0 \ .$$

Clearly then $\psi = \exp(-a|x|^2)\phi$ is a generalized eigenvector of K with the eigenvalue $\tilde{E}$. But then obviously ψ is an eigenvector (because K is self-adjoint). It follows that $\tilde{E} = E$ because the eigenspace $\mathrm{Ker}\,(K - E)$ already has dimension m, hence no other generalized eigenvectors (except the eigenvectors with the eigenvalue E) exist for the eigenvalues $\tilde{E}$ with $|\tilde{E} - E| < \varepsilon$. Also the coincidence of the dimensions implies that all eigenvectors ψ of K with the eigenvalue E will appear in this way. Since $\phi \in L^2(\mathbf{R}^n) \otimes \mathbf{C}^k$, the estimate (A.2) immediately follows. □

Remark. The argument that we gave to prove Proposition A.3, is close to the arguments used in the papers by A.O'Connor [Co], J.Combes and L.Thomas [C-T] (see also the description of the corresponding arguments in [R-S], Ch.XIII.11 and [Si1]). The arguments from [A] or [B-S] Ch.3 can be used too. A similar argument with the weight functions can be also used to prove decay estimates for Green functions of elliptic operators on manifolds (see [S5] and references there).

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