NOVIKOV INEQUALITIES FOR VECTOR FIELDS

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Introduction

A. Let M be a compact closed connected C^∞ manifold, $\dim M = n$. Let v be a C^∞ vector field on M. Consider a singular point of v, i.e. a point $\bar{x} \in M$ such that $v(\bar{x}) = 0$. In local coordinates x^i near $\bar{x}$ the vector field v can be written as

$$v(x) = \sum_{i=1}^{n} v^i(x) \frac{\partial}{\partial x^i} \ . \tag{0.1}$$

The singular point $\bar{x}$ is called *non-degenerate* if

$$\det \, [\partial v^i / \partial x^j](\bar{x}) \neq 0 \ . \tag{0.2}$$

A non-degenerate singular point is always isolated. The *index* of $\bar{x}$ is defined then as the sign of the determinant (0.2):

$$\operatorname{ind}_v(\bar{x}) = \operatorname{sign} \det \, [\partial v^i / \partial x^j](\bar{x}) \ . \tag{0.3}$$

(So the index is always ±1 in this case.)

Let us assume that all the singular points of v are non-degenerate. Then there is only a finite number of singular points. Denote by $m^\pm$ the number of the singular points with the index ±1. By the well known Poincaré-Hopf theorem

$$m^+ - m^- = \chi(M) \ , \tag{0.4}$$

where $\chi(M)$ is the Euler characteristic of M.

Denote by $\Lambda^p(M)$ the vector space of all C^∞ complex valued exterior p-forms on M and let $\Lambda^\bullet(M) = \oplus_{p=0}^n \Lambda^p(M)$ be the set of all exterior forms on M. Let also

$$\Lambda^+(M) = \bigoplus_k \Lambda^{2k}(M), \quad \Lambda^-(M) = \bigoplus_k \Lambda^{2k+1}(M) \ ,$$

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so $\Lambda^\bullet(M) = \Lambda^+(M) \oplus \Lambda^-(M)$. The exterior differential acts as follows:

$$d: \ \Lambda^p(M) \longrightarrow \Lambda^{p+1}(M) \ .$$

Denote by i_v the substitution operator of the vector field v into forms, so

$$i_v: \ \Lambda^p(M) \longrightarrow \Lambda^{p-1}(M) \ .$$

Combining d and i_v we obtain the deformed differentials

$$d_t = d + ti_v: \ \Lambda^\pm(M) \longrightarrow \Lambda^\mp(M) \ , \tag{0.5}$$

where t is a real parameter and the choice of signs will always be clear from the context. Obviously d_t is a first-order differential operator.

Let us choose a Riemannian metric g on M. This metric induces the volume element (a C^∞ positive density) on M and Hermitian inner products on all the spaces $\Lambda^p(M)$. Then we can define the adjoint operator to d_t which will be denoted by d_t^*, and also deformed Euler and Laplace operators

$$\partial_t^\pm = d_t + d_t^*: \ \Lambda^\pm(M) \longrightarrow \Lambda^\mp(M) \ , \tag{0.6}$$

$$\Delta^\pm(t) = (d_t + d_t^*)^2: \ \Lambda^\pm(M) \longrightarrow \Lambda^\pm(M) \ . \tag{0.7}$$

Clearly $\Delta^+(t) = \partial_t^- \partial_t^+$ and $\Delta^-(t) = \partial_t^+ \partial_t^-$.

We shall study the numbers

$$b^\pm(t) = \dim_{\mathbf{C}} \operatorname{Ker} \partial^\pm(t) = \dim_{\mathbf{C}} \operatorname{Ker} \Delta^\pm(t) \ . \tag{0.8}$$

For these numbers we have

$$b^+(t) - b^-(t) = \chi(M) \ . \tag{0.9}$$

Indeed, the left hand side of (0.9) is equal to the index of the elliptic differential operator ∂_t^+, i.e. to

$$\begin{aligned} \operatorname{ind} \ \partial_t^+ &= \dim \operatorname{Ker} \partial_t^+ - \dim \operatorname{Coker} \partial_t^+ = \dim \operatorname{Ker} \partial_t^+ - \dim \operatorname{Ker} \partial_t^- \\ &= \dim \operatorname{Ker} \Delta^+(t) - \dim \operatorname{Ker} \Delta^-(t) \ . \end{aligned} \tag{0.10}$$

But the index does not depend on the lower order terms of the elliptic operators; therefore it does not depend on t. Taking $t = 0$, we obtain the operator $d + d^*$ whose index is

$$\operatorname{ind} \, (d + d^*) = \dim_{\mathbf{C}} \bigoplus_k H^{2k}(M, \mathbf{C}) - \dim_{\mathbf{C}} \bigoplus_k H^{2k+1}(M, \mathbf{C}) = \chi(M) \ .$$

B. In Appendix to [N-S] S.P. Novikov sketched a proof of the following

Theorem 0.1. *There exists* $t_0 > 0$ *such that for any* $t > t_0$

$$m^{\pm} \geq b^{\pm}(t)\ . \tag{0.11}$$

This statement was known to E.Witten (it is implicitly contained on page 681 of [W1] as a side remark but no details are given).

The most important and unexpected element in the Novikov proof was the fermionic Bogolyubov transformation which allows us to diagonalize a $2^n \times 2^n$ matrix of a special form. This matrix appears as an element of the model operator obtained when we replace all entries of the operator $\Delta^{\pm}(t)$ by their "principal" parts near a singular point. The matrix represents an operator which is quadratic with respect to creation and annihilation operators in the fermionic Fock space (or, equivalently, in the vector space where an irreducible representation of the Clifford algebra is given). One of the goals of our paper is to explain this technique in more detail.

The substantiation of the replacing of the operator $\Delta^{\pm}(t)$ by its "principal" part was not discussed in [N-S]. This can be done in a very general frame of semi-classical asymptotics as explained in [S].

Novikov also conjectured in [N-S] that the numbers

$$b^{\pm} = \lim_{t\to\infty} b^{\pm}(t) \tag{0.12}$$

in fact do not depend on the Riemannian metric and are therefore invariants of the pair (M, v). M. Braverman gave a counterexample to this conjecture (see Appendix). However we shall explain that in a way a positive answer for the Novikov conjecture is possible if we restrict ourselves to an open dense set of Riemannian metrics.

Let us comment on the behavior of the numbers $b^{\pm}(t)$ as functions of t. It occurs that there exists a discrete set $S \subset \mathbf{R}$ such that

$$b^{\pm}(t) = \underline{b}^{\pm} = \underline{b}^{\pm}(v, g) = \text{const} \ \text{ if } \ t \in \mathbf{R} - S^{\pm}; \quad b^{\pm}(t) > \underline{b}^{\pm} \ \text{ if } \ t \in S^{\pm}\ . \tag{0.13}$$

This means that we can present S as an ordered sequence $\ldots < t_k < t_{k+1} < \ldots$ (possibly finite or empty), the function $b^{\pm}(t)$ is constant outside this sequence and has a removable singularity (a jump up) at each point t_k. The limit in (0.12) exists if and only if the jump points t_k do not go to $+\infty$ and in this case $b^{\pm} = \underline{b}^{\pm}$. If this is not the case we still can write numbers $\underline{b}^{\pm}$ in the form

$$\underline{b}^{\pm} = \lim_{t\to\infty} \inf b^{\pm}(t)\ . \tag{0.14}$$

Note also that

$$\underline{b}^{+} - \underline{b}^{-} = \chi(M) \tag{0.15}$$

due to (0.9).

For obvious reasons it is natural to call the numbers $\underline{b}^{\pm}$ the *background values* of the functions $b^{\pm}(t)$. They seem to be reasonable invariants to consider because they have a kind of stability. The following result can be considered a partial positive answer to the Novikov conjecture mentioned above.

Theorem 0.2. *For any fixed vector field v on M there exists a set $\mathcal{G}^0 = \mathcal{G}^0_v$ of Riemannian metrics on M such that*
(i) *$\mathcal{G}^0$ is open in C^1 topology and dense in C^∞ topology in the set $\mathcal{G}$ of all Riemannian metrics on M;*
(ii) *$\underline{b}^{\pm}(v,g) = \underline{b}^{\pm}(v)$ does not depend on g if $g \in \mathcal{G}^0$;*
(iii) *$\underline{b}^{\pm}(v,g) > \underline{b}^{\pm}(v)$ if $g \notin \mathcal{G}^0$.*

This means that for generic Riemannian metrics the background values $\underline{b}^{\pm}$ of the dimensions $b^{\pm}(t)$ do not depend on the choice of the metric (and take the minimal possible values) if the vector field v is kept fixed.

It follows from (0.11) that

$$m^{\pm} \geq \underline{b}^{\pm} \ . \tag{0.16}$$

It is also obvious from (0.11) that

$$m^{\pm} \geq \lim_{t\to\infty} \sup b^{\pm}(t) \ . \tag{0.17}$$

These inequalities are stronger provided we really have an infinite number of jumps near $+\infty$. However it is not clear when the number of jumps is infinite. In fact for all explicitly known examples the only jump happens at $t = 0$. This is the case e.g. for any vector field on the circle (see Appendix) and for any Killing vector field ([W1]).

Denote

$$b^+(M) = \sum_k b_{2k}(M) \ , \quad b^-(M) = \sum_k b_{2k+1}(M) \ , \tag{0.18}$$

where $b_p(M)$ is the p^{th} Betti number of M. Then $b^+(M) - b^-(M) = \chi(M)$. For any gradient vector field $v = \operatorname{grad} f$, $f \in C^\infty(M)$, we have $b^{\pm}(t) = b^{\pm}(M)$ for all $t \in \mathbf{R}$. In addition to the inequalities (0.16) we shall prove

Proposition 0.3. *For any vector field v and any Riemannian metric g the background values $\underline{b}^{\pm}(v,g)$ satisfy the inequalities*

$$\underline{b}^{\pm}(v,g) \leq b^{\pm}(M) \ . \tag{0.19}$$

This means that the maximum of $\underline{b}^{\pm}$ over all vector fields on M is attained on the gradient vector fields. It is also easy to figure out what are the minimum values of these numbers. Indeed, since the manifold M is connected, it is always possible to construct a vector field v on M such that it has all its singular points non-degenerate and with the same index (we can eliminate pairs of singular points with opposite indices without changing other singular points). Then one of the numbers $m^{\pm}$ vanishes and another is equal to $|\chi(M)|$ due to (0.4). But then (0.15) and (0.16) imply that $\underline{b}^{\pm} = m^{\pm}$; so one of the numbers $\underline{b}^{\pm}$ vanishes and another is equal to $|\chi(M)|$. These are obviously the minimal possible values of $\underline{b}^{\pm}$. Let us denote them by $\underline{b}^{\pm}_{min}$.

Theorem 0.4. *There exists a set $\mathcal{V}^0(M)$ of vector fields on M such that*

(i) $\mathcal{V}^0$ *is open in C^1 topology and dense in C^∞ topology in the set $\mathcal{V}$ of all vector fields on M;*

(ii) $\underline{b}^{\pm}(v) = \underline{b}^{\pm}_{min}$ *if $v \in \mathcal{V}^0$.*

We see that the background values $\underline{b}^{\pm}(v)$ take the minimal values which do not depend on the vector field for generic vector fields and generic metrics. This seems to show that the background values do not carry any essential information about the vector field if we consider generic metrics only. Therefore it is reasonable to consider all the metrics. For example we will have then a necessary condition that a vector field can be made a gradient field by a choice of a Riemannian metric:

$$\max\{\underline{b}^{\pm}(v,g) |\ g \in \mathcal{G}\} = b^{\pm}(M)\ . \tag{0.20}$$

More general possibility is to look at the numbers $\underline{b}^{\pm}(v,g)$ as functions of $g \in \mathcal{G}$ for a fixed vector field v. It is sufficient to consider one of these two functions because their difference is constant due to (0.15). Then the level sets of this function form an analytic stratification in the set of all Riemannian metrics; it has a finite number of strata corresponding to all possible values of the function. The set of all possible values of the function on the non-empty strata is a simplest invariant that can be extracted from this stratification. Other invariants can be possibly obtained by considering the topology of this stratification.

C. Let us give some additional history and references.

The story starts with the celebrated paper of E. Witten [W1]. He suggested to use the deformed de Rham complex (with the deformation given by a Morse function) for a new proof of the classical Morse inequalities. More details about the Witten's proof can be found in [C-F-K-S], [H1], [H-S1], [He], [S], [Si]. Later Witten [W2] applied his method to obtain holomorphic Morse inequalities. Another important version of holomorphic Morse inequalities was suggested by J.-P. Demailly (see [D1],[D2], [Siu], [B2], [G1], [G2]).

J.-M. Bismut [B1] applied the Witten method to give a new proof of the degenerate Morse-Bott inequalities (see also [H2] and [H-S2]). A. Pazhitnov [P] used the method to prove some of the Novikov inequalities for multivalued functions, i.e. closed 1-forms (these inequalities appeared in [N] and in an important particular case were improved by M. Farber [F]). The combinatorial version of the Witten's method was developed by by R. Forman in [Fo1], [Fo2].

The Witten deformation has been also applied to the consideration of combinatorial and analytic torsions in [B-Z1], [B-Z2], [B-F-K1], [B-F-K2], [B-F-K-M].

The Witten method was applied in [S] to give an alternative proof of the L^2 Morse inequalities (on regular covering manifolds) which were first proved in [N-S] by topological methods. More general Morse inequalities on manifolds of bounded geometry (but with an additional requirement of amenability) were proved by J. Roe [R] (who also used the Witten method). The paper [S] also introduces an L^2 version of the invariants $b^{\pm}(t)$ and contains a proof of the L^2 version of Theorem 0.1.

The operators $d_t,\ \partial_t^{\pm},\ \Delta^{\pm}(t)$ were also first introduced by Witten [W1] who mainly treated the case of the Killing vector fields but made a few remarks about general vector fields as well. In particular he suggested a way to use his method to prove the Poincaré-Hopf relation (0.4) (see A. El Soufi and X.P. Wang [E-W] for a rigorous treatment). Witten did not require the Killing vector fields to have only non-degenerate singular points (so singular submanifolds are allowed). He proved that for these fields the only jump may happen at $t = 0$ and besides $\underline{b}^{\pm}(v, g) = b^{\pm}(N)$ where N is the singular submanifold. In particular, if all the singular points of the vector field v (which is a Killing vector field with respect to the metric g) are non-degenerate, then $\underline{b}^{\pm}(v, g) = \underline{b}^{\pm}_{min}$, so all Killing vector fields are generic in the sense of Theorem 0.4. The inequality (0.19) is also present in [W1] for the Killing vector fields. Witten used it also for the degenerate case which leads to the estimates $b^{\pm}(N) \leq b^{\pm}(M)$ and the equality $\chi(M) = \chi(N)$ generalizing the inequalities (0.11) and the Poincaré-Hopf relation (0.4). He also calculated the signature of the manifold in terms of the singular submanifold N (see [E-W] for more details).

In Appendix to [N-S] Novikov starts with the deformation $d_t = d + t\lambda_\omega$ where λ_ω is the operator of the external multiplication (in $\Lambda^\bullet(M)$) by a 1-form ω which corresponds to the vector field v with the chosen Riemannian metric on M. This leads to the same operators $\partial_t^{\pm}$.

D. I am very grateful to M. Farber for numerous discussions which we had during my stay in IHES. He explained to me the idea of the background value of the dimension of the kernel of the analytic family of elliptic operators and also helped me clarify one of the details of the Novikov proof. I am also grateful to M. Braverman who allowed me to include his counterexample into the text of this paper, to M. Gromov for useful comments and to P. Kuchment for help with references.

1. Model operator

A. Let us recall a necessary result about semiclassical asymptotics, which is a particular case of a very general theorem from [S].

Let M be a compact closed manifold with a fixed positive C^∞ volume element (a positive C^∞ density) e.g. a Riemannian density corresponding to a given Riemannian metric on M, $\dim_{\mathbf{R}} M = n$. Suppose E is a C^∞ Hermitian vector bundle on M. The (complex) dimension of its fiber will be denoted by k. Denote by $C^\infty(M, E)$ the space of all C^∞ sections of E over M and by $L^2(M, E)$ the Hilbert space of all L^2 sections of E over

M. Let us consider an elliptic self-adjoint second order differential operator $H = H(t)$ on M which depends on a real parameter $t > 0$ and has the form

$$H = -t^{-1}A + B + tV : \ C^\infty(M, E) \longrightarrow C^\infty(M, E) \ . \tag{1.1}$$

Here $-A$ is a second order elliptic self-adjoint differential operator with a non-negative principal symbol (e.g. $A = -\Delta = dd^* + d^*d$ in $\Lambda^p(M)$, $\Lambda^\bullet(M)$ or in $\Lambda^\pm(M)$), B and V are self-adjoint zero order operators i.e. algebraic morphisms of the bundle E. The operator V will be called *potential* and the whole operator H will be called the *hamiltonian*. We shall assume that V satisfies the following condition

(C): *$V(x) \geq 0$ for all $x \in M$. If at a point $\bar{x} \in M$ the matrix $V(\bar{x})$ degenerates then $V(\bar{x}) = 0$ and*

$$V(x) \geq c|x - \bar{x}|^2 I \text{ in a neighborhood of } \bar{x}. \tag{1.2}$$

Here I is the identity morphism of the corresponding fiber and the inequality is understood as the inequality of quadratic forms.

A point $\bar{x}$ with $V(\bar{x}) = 0$ will be called a *singular point.*

It follows that all the singular points are isolated. Let $\bar{x}$ be one of them. We want to form a model operator which will have a relatively simple form: a matrix harmonic oscillator which is as close to H near $\bar{x}$ as possible.

Suppose we have chosen local coordinates $x^1, \ldots, x^n$ and a trivialization of E near $\bar{x}$. For simplicity we will assume that the given volume element coincides with the Lebesgue volume element at the point $\bar{x}$. Then H becomes a $k \times k$ matrix differential operator. The operator A becomes an operator of the form

$$A = \sum_{1 \leq r,s \leq n} A_{rs}(x) \frac{\partial^2}{\partial x^r \partial x^s} + A^{(1)} \ ,$$

where $A^{(1)}$ has the order ≤ 1. Denote

$$A^{(2)} = \sum_{1 \leq r,s \leq n} A_{rs}(\bar{x}) \frac{\partial^2}{\partial x^r \partial x^s} \ ,$$

i.e. $A^{(2)}$ is the second order homogeneous matrix differential operator in $\mathbf{R}^n$ obtained from A by taking its higher order terms with the coefficients frozen at the point $\bar{x}$.

Let $\bar{B} = B(\bar{x})$, so $\bar{B}$ is an endomorphism of the fiber of the bundle E over the point $\bar{x}$. Hence $\bar{B}$ is just a $k \times k$ Hermitian matrix in the chosen trivialization of E.

Let us also define

$$V^{(2)}(x) = \frac{1}{2} \sum_{1 \leq r,s \leq n} \frac{\partial^2 V}{\partial x^r \partial x^s}(\bar{x}) x^r x^s \ ,$$

i.e. $V^{(2)}$ is the quadratic part of the potential V near $\bar{x}$.

Definition 1.1. The *model operator* of the hamiltonian H at the singular point $\bar{x}$ is the operator

$$K(\bar{x}) = -A^{(2)} + \bar{B} + V^{(2)}(x) \ , \tag{1.2}$$

which is a second order differential operator with polynomial coefficients in $\mathbf{R}^n$. (The coefficients by the derivatives are in fact constant, and the only variable coefficients are quadratic.)

Since both operators $-A^{(2)}$ and $V^{(2)}(x)$ are non-negative, the operator $K(\bar{x})$ is bounded from below. Due to the ellipticity of A and non-degeneracy condition (C) the operator $K(\bar{x})$ has a discrete spectrum.

Let $\bar{x}_1, \ldots, \bar{x}_N$ be the list of all singular points. Let us form the *model operator* for the hamiltonian H on M as

$$K = \bigoplus_{j=1}^{N} K(\bar{x}_j) \ . \tag{1.3}$$

Denote the eigenvalues of K by

$$\mu_1 < \mu_2 < \mu_3 < \ldots$$

and their multiplicities by

$$p_1, p_2, p_3, \ldots$$

(so $p_1, p_2, p_3, \ldots$ are positive integers). The following result is the simplest version of a general theorem from [S] and a slight generalization of Theorem 11.1 from [C-F-K-S]:

Proposition 1.2. *The eigenvalues of the hamiltonian H concentrate near the eigenvalues of the model operator K in the following sense: for any $q = 1, 2, \ldots$ there exist $t_0 > 0$ and $C > 0$ such that for any $t > t_0$*

(a) *there are precisely p_j eigenvalues (multiplicities counted) of the hamiltonian H in the interval $(\mu_j - Ct^{-1/5}, \mu_j + Ct^{-1/5})$, $j = 1, \ldots, q$;*

(b) *there are no eigenvalues of H in $(-\infty, \mu_1 - Ct^{-1/5})$ and in the intervals $(\mu_j + Ct^{-1/5}, \mu_{j+1} - Ct^{-1/5})$, $j = 1, \ldots, q$.*

Corollary 1.3. *Denote $l_j(t) = \dim \operatorname{Ker} (H(t) - \mu_j I)$. Then for any $j = 1, 2, \ldots$ there exists $t_0 > 0$ such that for any $t > t_0$*

$$l_j(t) \le p_j \ . \tag{1.4}$$

This corollary shows that we can estimate from above the dimensions of the eigenspaces of H provided we know explicitly the eigenvalues of the model operator. In the classical example of the Morse inequalities treated by the Witten approach, we should take H to be the deformed Laplacian on p-forms, $H = d_t d_t^* + d_t^* d_t$, where $d_t = d + t\lambda_{df}$, f is a Morse function on M, λ_{df} is the exterior multiplication operator by the 1-form df. Considering then $\mu_1 = 0$ we obtain $l_1(t) = b_p = \text{const}$ (here b_p is the Betti number) and p_1 becomes equal to m_p, the number of the critical points of f with the index p. Therefore in this case (1.4) gives the classical Morse inequalities. (Some simple additional arguments allow us to obtain the stronger inequalities with alternating sums as well.)

B. Let us rewrite the deformed Laplacians $\Delta^{\pm}(t)$ from (0.7) in a more explicit form to check whether we can apply Proposition 1.2 and Corollary 1.3.

Note first that

$$i_v^* = \lambda_\omega \ , \tag{1.5}$$

where ω is the 1-form corresponding to the vector field v by the chosen Riemannian metric (in local coordinates the components ω_i of the form ω are given by $\omega_i = \sum_j g_{ij} v^j$).

We shall use the notation $\{A, B\} = AB + BA$ (the anti-commutator of the operators A and B).

Then using the vanishing of the squares of all operators $d, d^*, i_v, \lambda_\omega$ we obtain

$$\begin{aligned}\Delta^{\pm}(t) &= (d + d^* + t i_v + t\lambda_\omega)^2 \\ &= \{d, d^*\} + t[\{d, i_v\} + \{d^*, \lambda_\omega\} + \{d, \lambda_\omega\} + \{d^*, i_v\}] + t^2\{i_v, \lambda_\omega\} \ .\end{aligned}$$

We have $\{d, d^*\} = \Delta^{\pm}$ (the usual Laplacian on forms of even or odd degree). Also $\{d, i_v\} = \mathcal{L}_v$ is the Lie derivative of forms with respect to the given vector field v. It follows that $\{d^*, \lambda_\omega\} = \mathcal{L}_v^*$. An obvious calculation shows that $\{d, \lambda_\omega\} = \lambda_{d\omega}$ (the exterior multiplication by the 2-form $d\omega$), so $\{d^*, i_v\} = \lambda_{d\omega}^*$. Now an easy calculation shows that $\{i_v, \lambda_\omega\} = |v(x)|^2$ (the multiplication operator by the scalar function $x \mapsto |v(x)|^2$ where the absolute value means the norm of $v(x)$ in the given Riemannian metric). Therefore

$$\Delta^{\pm}(t) = \Delta^{\pm} + t(\mathcal{L}_v + \mathcal{L}_v^* + \lambda_{d\omega} + \lambda_{d\omega}^*) + t^2 |v(x)|^2 \ . \tag{1.6}$$

Let us consider the hamiltonian

$$H = H^{\pm} = H^{\pm}(t) = t^{-1}\Delta^{\pm}(t) \ . \tag{1.7}$$

It has the form (1.1) with

$$A = -\Delta^{\pm}, \quad B = \mathcal{L}_v + \mathcal{L}_v^* + \lambda_{d\omega} + \lambda_{d\omega}^* \quad \text{and} \quad V(x) = |v(x)|^2 \ . \tag{1.8}$$

Let us check the applicability of Theorem 1.2 (and Corollary 1.3) and calculate the model operator. We shall follow [C-F-K-S] where a similar calculation was made in the case where v is the gradient vector field of a Morse function, or, equivalently, if the form ω is exact.

We shall argue in a coordinate neighborhood $\mathcal{U}$ with local coordinates $x^1, \ldots, x^n$. We can (and will) always assume that the coordinates are chosen so that the Riemannian volume element coincides with the Lebesgue volume element in the given coordinates. Let us introduce the fermionic creation and annihilation operators a^{*i}, a^i acting in $\Lambda^\bullet(\mathcal{U})$ by the formulas

$$a^{*i} = \lambda_{dx^i}, \quad a^i = (a^{*i})^* = \sum_{j=1}^{n} g^{ij} i_{\partial/\partial x^j} \ . \tag{1.9}$$

They satisfy the following anti-commutation relations

$$\{a^i, a^j\} = \{a^{*i}, a^{*j}\} = 0, \quad \{a^i, a^{*j}\} = g^{ij}, \quad i, j = 1, \ldots, n. \tag{1.10}$$

Let us also introduce operators $\partial_i : \Lambda^\bullet(\mathcal{U}) \to \Lambda^\bullet(\mathcal{U})$ which map $\Lambda^p(\mathcal{U})$ into itself by differentiating all coefficients with respect to x^i:

$$\partial_i \sum_{i_1,\ldots,i_p} \omega_{i_1 i_2 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} = \sum_{i_1,\ldots,i_p} \frac{\partial \omega_{i_1 i_2 \ldots i_p}}{\partial x^i} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \ . \tag{1.11}$$

Clearly ∂_i commutes with a^{*j}, hence ∂_i^* commutes with a^j. Note also that

$$\partial_i^* = -\partial_i + A_i, \quad \text{ord}\, A_i = 0 \ . \tag{1.12}$$

The exterior differential d can be written in the form

$$d = \sum_i a^{*i} \partial_i \ . \tag{1.13}$$

Therefore

$$d^* = \sum_i a^i \partial_i^* = -\sum_i a^i \partial_i + \sum_i a^i A_i \ . \tag{1.14}$$

For the given vector field v and the corresponding 1-form ω we have

$$\lambda_\omega = \sum_i \omega_i a^{*i}, \quad i_v = \sum_i \omega_i a^i = \sum_{i,j} g_{ij} v^i a^j. \tag{1.15}$$

Lemma 1.4.

$$\text{ord}(\mathcal{L}_v + \mathcal{L}_v^*) = 0 \ . \tag{1.16}$$

Proof. We have

$$\mathcal{L}_v = \{d, i_v\} = \{\sum_i a^{*i} \partial_i, \sum_j \omega_j a^j\} = \sum_{i,j} \{a^{*i}, a^j\} \omega_j \partial_i + R_0 = \sum_{i,j} g^{ij} \omega_j \partial_i + R_0 \ ,$$

where R_0 is a zero order operator. It follows that

$$\mathcal{L}_v^* = -\sum_{i,j} g^{ij}\omega_j\partial_i + R_0'$$

with another zero order operator R_0'. Then (1.16) immediately follows. $\square$

Corollary 1.5. *In* (1.8) *we have* ord $B = 0$.

Corollary 1.6. *Let all the singular points of the vector field v be non-degenerate. Then the operators $t^{-1}\Delta^{\pm}(t)$ have the form* (1.1) *and satisfy all the conditions required to apply Proposition 1.2 and Corollary 1.3.*

C. Let us calculate the model operator $K(\bar{x})$ for the hamiltonians $t^{-1}\Delta^{\pm}(t)$ at a singular point $\bar{x}$. To do this we will use special coordinates near $\bar{x}$, chosen in the same way as in [C-F-K-S]. Namely we will assume that these coordinates are orthogonal at $\bar{x}$ up to second order i.e.

$$g_{ij}(\bar{x}) = \delta_{ij} \quad \text{and} \quad \frac{\partial g_{ij}}{\partial x^k}(\bar{x}) = 0 \ , \tag{1.17}$$

and also that the Riemannian volume element in these coordinates coincides with the Lebesgue volume element of the given coordinates. We will also assume that $\bar{x}$ corresponds to the origin of the new coordinates. If all these conditions are fulfilled we shall say that the coordinates are *canonical.* All further calculations of the model operator will be conducted in canonical coordinates. Note that in canonical coordinates

$$g_{ij} = \delta_{ij} + O(|x|^2) \quad \text{as} \quad |x| \to 0 \ . \tag{1.18}$$

For the calculation of the model operator we should replace A by the flat Laplacian acting componentwise. We should also replace B by $\bar{B} = B(\bar{x})$ and $V(x)$ by its quadratic part near 0. In both cases this implies that in calculations we can replace the vector field v by its linear part at $\bar{x} = 0$. Denote this linear part Cx where C is a real $n \times n$ matrix with the entries c_{ij}. We have then

$$v^i(x) = \sum_j c_{ij}x^j + O(|x|^2), \ \ \omega_i(x) = \sum_j c_{ij}x^j + O(|x|^2) \ ; \ \ c_{ij} = \frac{\partial v^i}{\partial x^j}(0) = \frac{\partial \omega^i}{\partial x^j}(0) \ . \tag{1.19}$$

Lemma 1.7. *In canonical coordinates near $\bar{x}$ we have*

$$\bar{B} = \sum_{i,j} c_{ij}(a^{*j} - a^j)(a^{*i} + a^i) \ . \tag{1.20}$$

Proof. We have to make our calculation of B above more precise to specify 0-order terms at the point $\bar{x} = 0$. First note that

$$i_v = \sum_i v^i a^i + O(|x|^2) = \sum_{i,j} c_{ij}x^j a^i + O(|x|^2)$$

and

$$i_v^* = \lambda_\omega = \sum_i \omega_i a^{*i} = \sum_{i,j} c_{ij} x^j a^{*i} + O(|x|^2)\ .$$

The terms $O(|x|^2)$ mean zero order operators which are $O(|x|^2)$ as $x \to 0$. They will not make any contributions in the model operator. We also have due to (1.18)

$$[\partial_i, a^j] = O(|x|)\ ,$$

where $O(|x|)$ is a zero order operator vanishing at the origin.

Now we can calculate $\mathcal{L}_v$:

$$\mathcal{L}_v = \{d, i_v\} = \{\sum_k a^{*k}\partial_k, \sum_{i,j} c_{ij} x^j a^i + O(|x|^2)\}$$

$$= \sum_{i,j,k} c_{ij} a^{*k} a^i \partial_k x^j + \sum_{i,j,k} c_{ij} a^i a^{*k} x^j \partial_k + O(|x|) + O_1(|x|^2)$$

$$= \sum_{i,j} c_{ij} a^{*j} a^i + \sum_{i,j,k} c_{ij} \{a^i, a^{*k}\} x^j \partial_k + O(|x|) + O_1(|x|^2)$$

$$= \sum_{i,j} c_{ij} a^{*j} a^i + \sum_{i,j} c_{ij} x^j \partial_i + O(|x|) + O_1(|x|^2)\ .$$

(Here terms $O_1(|x|^2)$ denote first-order operators whose coefficients are $O(|x|^2)$ as $x \to 0$.) Therefore, using (1.12) we obtain

$$\mathcal{L}_v^* = \sum_{i,j} c_{ij} a^{*i} a^j + \sum_{i,j} c_{ij} \partial_i^* x^j + O(|x|) + O_1(|x|^2)$$

$$= \sum_{i,j} c_{ij} a^{*i} a^j - \sum_{i,j} c_{ij} \partial_i x^j + O(|x|) + O_1(|x|^2)$$

$$= \sum_{i,j} c_{ij} a^{*i} a^j - \sum_i c_{ii} - \sum_{i,j} c_{ij} x^j \partial_i + O(|x|) + O_1(|x|^2)\ .$$

Since

$$\sum_i c_{ii} = \sum_{i,j} c_{ij} \delta_{ij} = \sum_{i,j} c_{ij} \{a^{*i}, a^j\} + O(|x|)\ ,$$

we get

$$\mathcal{L}_v^* = -\sum_{i,j} c_{ij} a^j a^{*i} - \sum_{i,j} c_{ij} x^j \partial_i + O(|x|) + O_1(|x|^2)\ .$$

Therefore

$$\mathcal{L}_v + \mathcal{L}_v^* = \sum_{i,j} c_{ij} (a^{*j} a^i - a^j a^{*i}) + O(|x|)\ . \tag{1.21}$$

(The terms $O_1(|x|^2)$ cancel due to Lemma 1.4.)

Let us turn to the other terms in B. Using (1.19) we get in our coordinates

$$d\omega = \sum_{i,j} c_{ij} dx^j \wedge dx^i + O(|x|) ,$$

hence

$$\lambda_{d\omega} = \sum_{i,j} c_{ij} a^{*j} a^{*i} + O(|x|) , \tag{1.22}$$

and

$$\lambda^*_{d\omega} = \sum_{i,j} c_{ij} a^i a^j + O(|x|) = -\sum_{i,j} c_{ij} a^j a^i + O(|x|) . \tag{1.22$'$}$$

Adding up (1.21), (1.22) and (1.22′) we get

$$B = \sum_{i,j} c_{ij}(a^{*j}a^i - a^j a^{*i} + a^{*j}a^{*i} - a^j a^i) + O(|x|) = \sum_{i,j} c_{ij}(a^{*j} - a^j)(a^{*i} + a^i) + O(|x|) ,$$

which immediately implies (1.20). □

It remains to write explicitly the quadratic part of $V(x)$ near $\bar{x}$. It has the form

$$V^{(2)}(x) = |Cx|^2 = (Cx, Cx) = (C^*Cx, x) . \tag{1.23}$$

or in coordinates

$$V^{(2)}(x) = \sum_{i,k,l} c_{ik} c_{il} x^k x^l . \tag{1.23$'$}$$

Summarizing all these calculations we see that we have proved the following

Proposition 1.8. *In canonical coordinates near a non-degenerate singular point $\bar{x}$ the model operator $K(\bar{x})$ has the form*

$$K(\bar{x}) = -\Delta + \sum_{i,j} c_{ij}(a^{*j} - a^j)(a^{*i} + a^i) + \sum_{i,k,l} c_{ik} c_{il} x^k x^l , \tag{1.24}$$

where $\Delta = \sum_i \partial_i^2$ and c_{ij} are the entries of the Jacobi matrix of v at $\bar{x}$ defined in (1.19).

2. Eigenvalues of the model operator

A. Our next goal is to calculate the eigenvalues of the model operator $K(\bar{x})$ of the form given in the Proposition 1.8. We shall consider this operator in the space of real (instead of complex) forms. Obviously this does not affect the eigenvalues. So all the functions and forms in this section will be real. Let us split $K(\bar{x})$ into the sum

$$K(\bar{x}) = K_0 + \bar{B}; \quad K_0 = -\Delta + V^{(2)}(x) = -\Delta + \sum_{i,k,l} c_{ik} c_{il} x^k x^l \ . \tag{2.1}$$

The operator $K(\bar{x})$ acts in one of the two spaces of vector functions; these spaces can be represented as the tensor products $L^2(\mathbf{R}^n) \otimes \Lambda^{\pm}$ where Λ^+, Λ^- are vector spaces of forms of even (respectively odd) degree at the origin in $\mathbf{R}^n$, $\dim_{\mathbf{R}} \Lambda^{\pm} = 2^{n-1}$. In this representation the operator K_0 has the form $\bar{K}_0 \otimes I$ where $\bar{K}_0$ acts by the same formula as K_0 (but on scalar functions instead of vector functions) and I is the identity operator on $\Lambda^{\pm}$. On the other hand the operator $\bar{B}$ there has the form $I \otimes \bar{B}_0$ where $\bar{B}_0$ is an operator on $\Lambda^{\pm}$. It follows that K_0 and $\bar{B}$ commute. Moreover if ϕ, χ are eigenvectors of $\bar{K}_0$, $\bar{B}_0$ with eigenvalues μ, ν respectively, then $\psi = \phi \otimes \chi$ is an eigenvector of $K(\bar{x})$ with the eigenvalue $\mu + \nu$. Besides if ϕ, χ run independently through orthonormal bases in $L^2(\mathbf{R}^n)$ and $\Lambda^{\pm}$ then the vectors $\phi \otimes \chi$ run through an orthonormal basis in $L^2(\mathbf{R}^n) \otimes \Lambda^{\pm}$. We come to the conclusion that the spectrum of $K(\bar{x})$ is just the arithmetic sum of the spectra of $\bar{K}_0$ and $\bar{B}_0$:

$$\operatorname{spec}(K(\bar{x})) = \operatorname{spec}(\bar{K}_0) + \operatorname{spec}(\bar{B}_0) = \{\mu + \nu |\ \mu \in \operatorname{spec}(\bar{K}_0),\ \nu \in \operatorname{spec}(\bar{B}_0)\} \ . \tag{2.2}$$

Therefore the calculation of the eigenvalues of $K(\bar{x})$ reduces to the calculation of the eigenvalues of $\bar{K}_0$ and $\bar{B}_0$.

Let us start with the calculation of the eigenvalues of $\bar{K}_0$. Denote by $s_1, \ldots s_n$ the eigenvalues of the matrix $|C| = \sqrt{C^t C}$, where C is the matrix with the entries c_{ij} defined by (1.19), C^t is the matrix transposed to C. Obviously $s_i > 0,\ i = 1, \ldots, n$. The numbers s_i are also called the *singular numbers* of the matrix C.

Lemma 2.1. *The eigenvalues of* $\bar{K}_0$ *are*

$$\sum_{i=1}^{n} (2k_i + 1) s_i \ ,\ k_i = 0, 1, 2, \ldots \ . \tag{2.3}$$

Proof. By an orthogonal transformation of the coordinates $x^1, \ldots, x^n$ we can diagonalize the quadratic form $V^{(2)}(x) = (C^t C x, x)$ i.e. replace it by the form

$$\sum_{i=1}^{n} s_i^2 (x^i)^2 \ .$$

This implies that we have the separation of variables for $\bar{K}_0$ and its eigenvalues are equal to sums of eigenvalues of the harmonic oscillators

$$-\frac{d^2}{d\xi^2} + s_i^2 \xi^2 \ \text{ in } \ L^2(\mathbf{R}) \ , \quad i = 1, \ldots, n.$$

Since these eigenvalues are $(2k_i + 1)s_i$, $k_i = 0, 1, 2, \dots$ (see e.g. [B-S], [G-J]), we obtain (2.3). $\square$

B. To calculate the eigenvalues of $\bar{B}_0$ we shall need a formalism of the fermionic Fock space. It is actually equivalent to the formalism of the representations of the Clifford algebras but we prefer the physical language as more suggestive.

Definition 2.3. The *fermionic Fock space with* n *degrees of freedom* is a finite-dimensional real euclidean vector space $\mathbf{F_n}$ with linear operators b^i, b^{*i}, $i = 1, \dots, n$, acting in it, so that the following requirements are fulfilled:

(i) b^i and b^{*i} are adjoint to each other with respect to the given inner product in $\mathbf{F_n}$;

(ii) the following anti-commutation relations are fulfilled:

$$\{b^i, b^j\} = \{b^{*i}, b^{*j}\} = 0, \quad \{b^i, b^{*j}\} = \delta_{ij}, \quad i, j = 1, \dots, n; \tag{2.4}$$

(iii) the space $\mathbf{F_n}$ is minimal in the sense that it does not contain any non-trivial subspace which is invariant under all the operators b^i, b^{*i}, $i = 1, \dots, n$.

The operators b^{*i} and b^i are called the fermionic *creation and annihilation operators* respectively.

Example 2.4. The space $\Lambda^\bullet$ of all differential forms at the origin of $\mathbf{R}^n$ with the operators $b^i = a^i, b^{*i} = a^{*i}$ is a fermionic Fock space with n degrees of freedom. Obviously $\dim \Lambda^\bullet = 2^n$ and $\Lambda^\bullet = \Lambda^+ \oplus \Lambda^-$.

We shall see that all the fermionic Fock spaces are isomorphic, so in fact they are all isomorphic to $\Lambda^\bullet$. However the abstract formalism plays a very important role in the calculation of the eigenvalues.

Definition 2.5. A *vacuum* or *vacuum vector* in $\mathbf{F_n}$ is a vector $\psi_0 \in \mathbf{F_n}$ such that $\|\psi_0\| = 1$ and

$$b^i \psi_0 = 0, \quad i = 1, \dots, n. \tag{2.5}$$

The vacuum in $\Lambda^\bullet$ is just the 0-form 1 (or -1).

Proposition 2.6. (a) *The vacuum in* $\mathbf{F_n}$ *exists and is unique up to a scalar factor* ± 1;

(b) *if* ψ_0 *is a vacuum vector in* $\mathbf{F_n}$ *then the vectors*

$$b^{*i_1} b^{*i_2} \dots b^{*i_k} \psi_0, \quad 1 \le i_1 < i_2 < \dots < i_k \le n \tag{2.6}$$

form an orthonormal basis in $\mathbf{F_n}$;

(c) *the action of the annihilation operator* b^i *on the basis vectors* (2.6) *is given by the formulas*

$$b^i (b^{*i_1} b^{*i_2} \dots b^{*i_k} \psi_0) = 0 \ \text{ if } \ i \notin \{i_1, \dots, i_k\} \,. \tag{2.7}$$

$$(2.7') \qquad b^{i_p}(b^{*i_1}b^{*i_2}\dots b^{*i_k}\psi_0) = (-1)^{p-1}(b^{*i_1}\dots b^{*i_{p-1}}b^{*i_{p+1}}\dots b^{*i_k}\psi_0)\ .$$

Proof. 1) Let us prove the existence of the vacuum first. It follows from the anti-commutation relations (2.4) that $(b^i)^2 = 0$. Therefore Ker $b^i \neq \{0\}$. Furthermore, it also follows from (2.4) that Ker b^n is invariant under the operators $b^1, \dots, b^{n-1}$. Arguing by induction we conclude that Ker $b^1 \cap$ Ker $b^2 \cap \dots \cap$ Ker $b^n \neq \{0\}$ and we can take any normalized vector from this intersection as the vacuum vector ψ_0.

2) The relations (2.7), (2.7′) easily follow from (2.4) and (2.5) if we move the operator b^i (or b^{i_p}) to the right using the relations (2.4), until it meets ψ_0 (or b^{*i_p} respectively).

3) To prove (b) we can write

$$(b^{*i_1}b^{*i_2}\dots b^{*i_k}\psi_0, b^{*j_1}b^{*j_2}\dots b^{*j_l}\psi_0) = (b^{*i_2}\dots b^{*i_k}\psi_0, b^{i_1}b^{*j_1}b^{*j_2}\dots b^{*j_l}\psi_0)\ ,$$

use (2.7), (2.7′) and then argue by induction with respect to $\max\{k, l\}$.

4) To prove the uniqueness of the vacuum vector let us argue by contradiction and assume that there exist two orthogonal vacuum vectors. Let us form subspaces L_1 and L_2 (in $\mathbf{F_n}$) spanned by the vectors of the form (2.6) for each of these vacuum vectors. Then both L_1 and L_2 are invariant with respect to all the operators b^i, b^{*i}. Now the same arguments as in 3) show that L_1 and L_2 are orthogonal which contradicts to the minimality of $\mathbf{F_n}$. □

Corollary 2.7. *The fermionic Fock space with n degrees of freedom is unique up to an isometric isomorphism.*

Proof. The desired isomorphism between two such spaces can be obtained by mapping the vectors of the basis (2.6) of the first space to the corresponding vectors in the second space. □

C. Our next tool will be Bogolyubov transformations (automorphisms of the Clifford algebra) which transform the structure of the fermionic Fock space by linear transformations of annihilation and creation operators. Suppose we are given a fermionic Fock space $\mathbf{F_n}$ with the annihilation and creation operators b^i, b^{*i}. We shall not fix the vacuum vector which is defined by the given operators b^i, b^{*i} up to the factor ±1.

Definition 2.8. *Bogolyubov transformation* of $\mathbf{F_n}$ is a set of new annihilation and creation operators a^i, a^{*i}, $i = 1, \dots, n$ in $\mathbf{F_n}$ which satisfy the usual fermionic anti-commutation relations (2.4) (with b replaced by a) and are connected with the operators b^i, b^{*i} by the formulas

$$(2.8) \qquad a^i = \sum_j (t^i_j b^j + p^i_j b^{*j}), \quad a^{*i} = \sum_j (p^i_j b^j + t^i_j b^{*j})\ ,$$

where $T = (t^i_j)$ and $P = (p^i_j)$ are real $n \times n$ matrices.

We shall later describe all possible matrices T and P and in particular prove the invertibility of any Bogolyubov transformation.

It is more convenient to work in matrix notations. Let a, a^*, b, b^* be columns of the height n with the entries a^i, a^{*i}, b^i, b^{*i} respectively, $i = 1, \dots, n$. We shall use the notations $a^t, a^{*t}, \dots$ for the rows which are transposed to the columns $a, a^*, \dots$. The anti-commutation relations (2.4) for a, a^* can be rewritten as

$$\{a, a^t\} = \{a^*, a^{*t}\} = 0, \quad \{a, a^{*t}\} = I \ , \tag{2.9}$$

where I is the $n \times n$ identity matrix, the notation $\{A, B\}$ for two matrices with operator entries mean the usual product of these matrices where the multiplication of the elements is replaced by the anti-commutator i.e.

$$\{A, B\}_{ij} = \sum_k \{a_{ik}, b_{kj}\} \ .$$

The transformation (2.8) can be rewritten in the matrix form

$$a = Tb + Pb^*, \quad a^* = Pb + Tb^* \ . \tag{2.10}$$

Lemma 2.9. *Let us assume that the operators* b^i, b^{*i} *satisfy the canonical anti-commutation relations* (2.4). *Then*

(i)*the operators* a^i, a^{*i} *satisfy the same relations* (*e.g.* (2.9)) *if and only if the matrices*

$$U = T - P, \quad V = T + P \tag{2.11}$$

are orthogonal i.e. $UU^t = VV^t = I$*;*

(ii) *if this is the case, then the transformation* (2.10) *is invertible and the inverse transformation is given by the formulas*

$$b = T^t a + P^t a^*, \quad b^* = P^t a + T^t a^* \ . \tag{2.12}$$

Proof. 1) We can rewrite (2.10) in the form

$$a + a^* = V(b + b^*), \quad a - a^* = U(b - b^*) \tag{2.13}$$

where U and V are given by (2.11). Denote $z = a + a^*$, $w = a - a^*$, so z and w are columns with the entries $z^i = a^i + a^{*i}$, $w^i = a^i - a^{*i}$. Denote also $\bar{z} = b + b^*$, $\bar{w} = b - b^*$. Then (2.13) can be rewritten in the form

$$z = V\bar{z}, \quad w = U\bar{w} \ . \tag{2.14}$$

It is easy to see that the canonical anti-commutation relations (2.9) for a, a^* are equivalent to the following anti-commutation relations for z, w:

$$\{z, z^t\} = 2I, \quad \{w, w^t\} = -2I, \quad \{z, w^t\} = \{w, z^t\} = 0 \ . \tag{2.15}$$

(In fact the equality $\{w, z^t\} = 0$ is equivalent to $\{z, w^t\} = 0$.) Similarly the relations (2.4) are equivalent to

$$\{\bar{z}, \bar{z}^t\} = 2I, \quad \{\bar{w}, \bar{w}^t\} = -2I, \quad \{\bar{z}, \bar{w}^t\} = \{\bar{w}, \bar{z}^t\} = 0 \; . \tag{2.16}$$

Assume that the relations (2.16) are satisfied. Then we obtain

$$\{z, w^t\} = \{V\bar{z}, \bar{w}^t U^t\} = 0$$

for any matrices V, U. Hence $\{w, z^t\} = 0$. We can also rewrite the first relation in (2.15) in the form

$$2I = \{z, z^t\} = \{V\bar{z}, \bar{z}^t V^t\} = 2VV^t \; ,$$

so it is satisfied if and only if the matrix V is orthogonal. Similarly the second relation in (2.15) is equivalent to the orthogonality of U. This proves (i).

2) To prove (ii) we should just note that for orthogonal matrices V and U (2.14) is equivalent to

$$\bar{z} = V^t z, \; \bar{w} = U^t w \; . \qquad \square$$

Corollary 2.10. *The general form of the fermionic Bogolyubov transformation* (2.10) *is obtained if we take*

$$T = \frac{1}{2}(U + V), \quad P = \frac{1}{2}(V - U), \tag{2.17}$$

where U and V are arbitrary orthogonal matrices.

Remark 2.11. Obviously the Bogolyubov transformations of $\mathbf{F_n}$ form a semigroup with respect to the obvious composition operation. It follows from Lemma 2.9 and formulas (2.14) that it is in fact a group which is isomorphic to $O(n) \times O(n)$.

D. In the standard Example 2.4 of the fermionic Fock space we actually have an important additional structure which has to be taken into account: the structure of a superspace.

Definition 2.12. *Fermionic Fock superspace with n degrees of freedom* is a fermionic Fock space $\mathbf{F_n}$ which has a splitting into a direct sum of vector spaces

$$\mathbf{F_n} = \mathbf{F_n^+} \oplus \mathbf{F_n^-} \; , \tag{2.18}$$

such that both annihilation and creation operators b^i, b^{*i} map $\mathbf{F_n^+}$ to $\mathbf{F_n^-}$ and $\mathbf{F_n^-}$ to $\mathbf{F_n^+}$.

In the standard Example 2.4 we naturally take $\mathbf{F_n^\pm} = \mathbf{\Lambda}^\pm$.

Using the standard language of supermathematics we shall say that the elements of $\mathbf{F_n^+}$ are *even* and the elements of $\mathbf{F_n^-}$ are *odd*. (There exist elements which are neither even nor odd, and the only element which is both even and odd is 0). Also a linear operator in a superspace is called *even* if it preserves parity of the elements and *odd* if it changes this parity. So the operators b^i and b^{*i} are odd.

Note that any Bogolyubov transformation of a fermionic Fock superspace preserves the superstructure, i.e. the operators a^i, a^{*i} also change parity.

Let us consider a vacuum vector ψ_0 in a fermionic Fock superspace $\mathbf{F_n}$.

Lemma 2.13. (i) *The vacuum in a fermionic Fock superspace is either even or odd, i.e. either* $\psi_0 \in \mathbf{F_n^+}$ *or* $\psi_0 \in \mathbf{F_n^-}$ *(but not both).*

(ii) *The splitting* (2.18) *is orthogonal and* $\dim \mathbf{F_n^+} = \dim \mathbf{F_n^-} = \mathbf{2^{n-1}}$.

Proof. 1) We can always write $\psi_0 = \psi_0^+ + \psi_0^-$ where $\psi_0^\pm \in \mathbf{F_n^\pm}$. Obviously both vectors $\psi_0^\pm$ satisfy the relations $b^i\psi_0^\pm = 0$. Due to the uniqueness of the vacuum (Proposition 2.6 (a)) one of the vectors $\psi_0^\pm$ should vanish. This proves (i).

2) Each element of the orthonormal basis (2.6) belongs either to $\mathbf{F_n^+}$ or to $\mathbf{F_n^-}$. The claim (ii) immediately follows. □

Definition 2.14. (i) A fermionic Fock superspace $\mathbf{F_n}$ is called *positive* if its vacuum is even (i.e. belongs to $\mathbf{F_n^+}$) and *negative* if its vacuum is odd (i.e. belongs to $\mathbf{F_n^-}$).

(ii) A Bogolyubov transformation of a fermionic Fock superspace is called *positive* if it preserves the parity of the vacuum and *negative* if it changes this parity.

The standard fermionic Fock superspace of the Example 2.4 is obviously positive.

Example 2.15. The identity transformation

$$a^i = b^i,\ a^{*i} = b^{*i},\quad i = 1, \dots, n,$$

is a positive Bogolyubov transformation corresponding to the matrices $V = U = I$.

Let us show that a Bogolyubov transformation really can be negative, i.e. can change the parity of the vacuum.

Example 2.16. Consider the transformation defined by

$$a^1 = b^{*1},\ a^2 = b^2,\ \dots,\ a^n = b^n\ . \tag{2.19}$$

Then necessarily

$$a^{*1} = b^1,\ a^{*2} = b^{*2},\ \dots,\ a^{*n} = b^{*n}\ ,$$

and it is obvious that this is a Bogolyubov transformation. Let ψ_0 be a vacuum for the operators b^i i.e. $\|\psi_0\| = 1$ and $b^i\psi_0 = 0,\ i = 1, \dots, n$. Then it is easy to see that $\phi_0 = b^{*1}\psi_0$ is a vacuum for the operators a^i. Obviously the parity of ϕ_0 is opposite to the parity of ψ_0, so the Bogolyubov transformation (2.19) is negative.

Using (2.13) we immediately see that for the transformation (2.19)

$$V = I,\quad U = \mathrm{diag}\{-1, 1, \dots, 1\}\ , \tag{2.20}$$

i.e. U is a reflection in the first variable.

Example 2.17. Let us take

$$a^1 = -b^{*1},\ a^2 = b^2,\ \ldots,\ a^n = b^n\ . \tag{2.21}$$

Then

$$a^{*1} = -b^1,\ a^{*2} = b^{*2},\ \ldots,\ a^{*n} = b^{*n}$$

and this is a Bogolyubov transformation. If ψ_0 is a vacuum for b^i then $\phi_0 = b^{*1}\psi_0$ is a vacuum for a^i, so again the transformation (2.21) is negative.

The matrices V and U corresponding to the transformation (2.21) are

$$V = \operatorname{diag}\{-1, 1, \ldots, 1\},\quad U = I\ . \tag{2.22}$$

Example 2.18. Let us take

$$a^1 = -b^1,\ a^2 = b^2,\ \ldots,\ a^n = b^n\ . \tag{2.23}$$

Then

$$a^{*1} = -b^{*1},\ a^{*2} = b^{*2},\ \ldots,\ a^{*n} = b^{*n}\ .$$

This is a positive Bogolyubov transformation (the vacuum does not change) with the corresponding matrices

$$V = U = \operatorname{diag}\{-1, 1, \ldots, 1\}\ . \tag{2.24}$$

Proposition 2.19. *A Bogolyubov transformation defined by the orthogonal matrices* V, U *is positive if and only if* $\det V \det U > 0$ *(and is negative if and only if* $\det V \det U < 0$*).*

Proof. Let us consider the vacuum up to a multiplication by ±1 as a point in $P^{n-1} = S^{n-1}/\mathbf{Z}_2$. Then it depends continuously on the pair $(V, U) \in O(n) \times O(n)$ because the system of linear equations for ψ_0 has a constant rank. Therefore the parity of the vacuum is locally constant, hence constant on each connected component of $O(n) \times O(n)$. But there are precisely 4 connected components in this group defined by the (continuous) map $(V, U) \to (\det V, \det U)$ from $O(n) \times O(n)$ to the 4-point set $\{\pm 1\} \times \{\pm 1\}$. Now the examples 2.15–2.18 provide transformations corresponding to 4 points in $O(n) \times O(n)$, exactly one in each connected component, and also show that the desired statement is true for these particular transformations. It follows by continuity that it is always true. $\square$

Remark. Another proof can be obtained if we present each matrix V, U as a product of reflections and use the fact that the parity of the vacuum (±1) defines a homomorphism of $O(n) \times O(n)$ to the multiplicative group $\{\pm 1\}$.

E. Let us return to the calculation of the eigenvalues of the operator

$$\bar{B}_0 = \sum_{i,j} c_{ij}(a^{*j} - a^j)(a^{*i} + a^i)\ , \tag{2.25}$$

acting in $\Lambda^\bullet$. Actually the operator $\bar{B}_0$ splits into an orthogonal direct sum

$$\bar{B}_0 = \bar{B}_0^+ \oplus \bar{B}_0^-\,, \qquad \bar{B}_0^\pm :\ \Lambda^\pm \to \Lambda^\pm\ . \tag{2.26}$$

(Note that earlier we wrote $\bar{B}_0$ instead of either of the operators $\bar{B}_0^\pm$.) However it is more convenient to start with the operator in the direct sum to have more freedom in the use of the operators a^i, a^{*i} which change the parity. As before we shall always assume that the matrix $C = [c_{ij}]$ is non-degenerate.

Proposition 2.20. *There exists a Bogolyubov transformation of the positive fermionic Fock superspace $\Lambda^\bullet$ with the new annihilation and creation operators b^i, b^{*i} such that*

(i) *the operator $\bar{B}_0$ takes the form similar to* (2.25) *but with a diagonal matrix C, more precisely*

$$\bar{B}_0 = \sum_{i=1}^{n} s_i(b^{*i} - b^i)(b^{*i} + b^i) = \sum_{i=1}^{n} s_i(2b^{*i}b^i - I)\ ; \tag{2.27}$$

(ii) *the parity of this Bogolyubov transformation* (± 1) *coincides with* sign det C *i.e. the new vacuum is in Λ^+ if* $\det C > 0$ *and in Λ^- if* $\det C < 0$.

Proof. We can rewrite (2.25) in the matrix form

$$\bar{B}_0 = (a^* - a)^t C^t (a^* + a)\ .$$

Substituting $a^* + a = V(b^* + b)$, $a^* - a = U(b^* - b)$ (hence $(a^* - a)^t = (b^* - b)^t U^t$) with orthogonal matrices V, U we see that

$$\bar{B}_0 = (b^* - b)^t U^t C^t V (b^* + b)\ ,$$

i.e. the matrix C is replaced by $V^t C U$. Let us take the polar decomposition $C = W|C|$ where $W \in O(n)$, $|C| = \sqrt{C^t C}$ and note that sign det W = sign det C. We can also diagonalize $|C|$ by an orthogonal transformation i.e. write $|C| = YSY^t$ where $Y \in O(n)$ and $S = \mathrm{diag}\{s_1, \dots, s_n\}$, $s_1, \dots, s_n$ are the eigenvalues of $|C|$. Then we get $V^t C U = V^t W Y S Y^t U$ and we can take $V = WY$ and $U = Y$ to assure that $V^t C U = S$. Hence (i) is proved and (ii) immediately follows from Proposition 2.19. □

Proposition 2.21. (i) *Eigenvalues of $\bar{B}_0$ in $\Lambda^\bullet$ are*

$$(s_{i_1} + \ldots + s_{i_k}) - (s_{i_{k+1}} + \ldots + s_{i_n})\ , \tag{2.28}$$

where $i_1 < \ldots < i_k$, $i_{k+1} < \ldots < i_n$ and $\{i_1, \dots, i_k\}$, $\{i_{k+1}, \dots, i_n\}$ are two complementary subsets of $\{1, 2, \dots, n\}$.

(ii) *The smallest eigenvalue $-s_1 - s_2 - \ldots - s_n$ corresponds to the vacuum of the operators b^i, and this vacuum is in Λ^+ if and only if* $\det C > 0$.

Proof. Denote by ψ_0 the vacuum for the operators b^i, $i = 1, \dots, n$, and consider the vector

$$b^{*i_1} b^{*i_2} \ldots b^{*i_k} \psi_0\ . \tag{2.29}$$

Using the formulas (2.7), (2.7′) we easily obtain that the vector (2.29) is an eigenvector for $2b^{*i}b^i - I$ with the eigenvalue 1 if $i \in \{i_1, \dots i_k\}$ and -1 otherwise. Therefore using the canonical form (2.27) we immediately come to the formula (2.28) for the eigenvalue of $\bar{B}_0$ corresponding to the same vector. This proves (i). The statement (ii) now follows from (ii) in the Proposition 2.20. □

Using (2.2) and Lemma 2.1 we come to

Corollary 2.22. (i) *The eigenvalues of* $K(\bar{x})$ *in* $L^2(\mathbf{R}^n) \otimes \Lambda^\bullet$ *are*

$$\sum_{i=1}^{n} (2k_i + 1)s_i + (s_{i_1} + \ldots + s_{i_k}) - (s_{i_{k+1}} + \ldots + s_{i_n}) \ , \tag{2.30}$$

where $k_i = 0, 1, \ldots$ *and the other notations are as in Proposition 2.21.*

(ii) 0 *is a simple eigenvalue of* $K(\bar{x})$ *and the corresponding eigenvector is a form of even degree if* $\det C > 0$ *and odd degree if* $\det C < 0$.

Returning to the model operators $K^\pm$ of the form (1.3) for the operators $t^{-1}\Delta^\pm(t)$ (see (1.6) and (1.7)) we come to

Corollary 2.23. *The multiplicity of* 0 *as an eigenvalue of the model operator* $K^\pm$ *is equal to* $m^\pm$.

Proof of Theorem 0.1. We can apply Corollary 1.3 to the operators $H^\pm(t) = t^{-1}\Delta^\pm(t)$, taking $\mu_1 = 0$. Then $l_1(t) = b^\pm(t)$ and Corollary 2.23 shows that $p_1 = m^\pm$, so Theorem 0.1 immediately follows. □

3. Background values of dimensions

A. Let us start with a well known result on the dimension of the kernel in a family of Fredholm operators. Let us recall that a linear bounded operator $A : \mathcal{B}_1 \to \mathcal{B}_2$ (here $\mathcal{B}_1, \mathcal{B}_2$ are complex Banach spaces) is called a *Fredholm* operator if $\dim \operatorname{Ker} A < \infty$ and $\dim \operatorname{Coker} A < \infty$ (then the image $\operatorname{Im} A$ is automatically closed).

Suppose that we have an *analytic family* $A(t) : \mathcal{B}_1 \to \mathcal{B}_2$ *of Fredholm operators* which is defined for $t \in \Omega$ where Ω is a connected complex manifold. This means that for each value of $t \in \Omega$ the operator $A(t)$ is Fredholm and the function $t \mapsto A(t)$ is analytic (holomorphic) as a function of $t \in \Omega$ with values in the Banach space of all bounded linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$. Then we have

Proposition 3.1. *There exists an analytic subset* $S \subset \Omega$, $S \neq \Omega$, *such that*

$$\dim \operatorname{Ker} A(t) = \bar{b} = \text{const} \ \text{ if } \ t \in \Omega - S \ \text{ and } \ \dim \operatorname{Ker} A(t) > \bar{b} \ \text{ if } \ t \in S \ . \tag{3.1}$$

This proposition is due to S.G. Krein and V.P. Trofimov and it is proved in [Z-K-K-P] (see also [G-K] for the case $\dim_{\mathbf{C}} \Omega = 1$). For more information about holomorphic Fredholm operator functions and further references see [Ku].

In case of Hilbert spaces $\mathcal{B}_1, \mathcal{B}_2$ more general semi-Fredholm operators (operators A with $\dim \operatorname{Ker} A < \infty$ and closed $\operatorname{Im} A$) can be considered instead of Fredholm ones in Proposition 3.1.

Definition 3.2. (i) We shall call S a *singular* or *exceptional set*, and the points $t \in S$ will be called *singular* or *exceptional values* of the parameter t.

(ii) $\bar{b}$ will be called the *background value* of the function $t \mapsto \dim \operatorname{Ker} A(t)$.

We will only deal with real-analytic families of Fredholm operators defined for $t \in X$ where X is a connected real-analytic manifold. The statement of Proposition 3.1 is still true for such a family because we can always extend it to a holomorphic family in a connected complex neighborhood of X. We shall consider only real singular set which is the intersection of the complex singular set with X. We will still denote the real singular set by S. It is a real-analytic subset in X, $S \neq X$. It follows that S is a closed nowhere dense subset in X.

An important particular case is $\dim_{\mathbf{R}} X = 1$, i.e. X is an open interval (possibly infinite of semi-infinite) in $\mathbf{R}$. In this case the singular set S is discrete. Hence we have the following

Corollary 3.3. *Suppose that $A(t)$ is a real-analytic family of Fredholm operators defined on an open interval $I \subset \mathbf{R}$ (possibly infinite or semi-infinite). There exists a discrete subset $S \subset I$ such that*

$$\dim \operatorname{Ker} A(t) = \bar{b} = \text{const} \;\text{ if }\; t \notin S \;\text{ and }\; \dim \operatorname{Ker} A(t) > \bar{b} \;\text{ if }\; t \in S\,. \tag{3.2}$$

Remark. If in this case we additionally assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Hilbert spaces, then replacing A by A^*A we can reduce the investigation of $\dim \operatorname{Ker} A(t)$ to the case when $A(t)$ is self-adjoint for all $t \in I$. This allows us to prove Corollary 3.3 with the help of the analytic perturbation technique (see e.g. Supplement 2 in [B-S], Ch. 7 in [K] or Theorem XII.13 in [R-S]). This method provides a more detailed information about the behavior of all eigenvalues of $A(t)$ and the corresponding eigenvectors.

Note also that the fact that jumps can be only up follows from the upper semicontinuity of the function $t \mapsto \dim \operatorname{Ker} A(t)$.

B. The results on analytic families of Fredholm operators can be applied to analytic families of elliptic operators if we consider them as Fredholm operators in appropriate Sobolev spaces. In particular we can consider the families $\Delta^{\pm}(t)$ (see (0.7) and (1.6)) which are quadratic polynomials with respect to t. For the dimensions of the kernels $b^{\pm}(t)$ (see (0.8)) we have

Proposition 3.4. *There exists a discrete subset $S \subset \mathbf{R}$ such that $b^{\pm}(t) = \underline{b}^{\pm} = \text{const}$ if $t \in \mathbf{R} - S$ and $b^{\pm}(t) > \underline{b}^{\pm}$ if $t \in S$.*

Proof. Due to (0.9) it is sufficient to consider one of the signs (and the singular set S will be the same for both functions $b^{\pm}(t)$).

For any integer $k \geq 0$ denote by W^k the standard Sobolev space on M (functions or sections of a vector bundle with all derivatives of order $\leq k$ in L^2). In particular $W^0 = L^2$. It will be clear from the context which vector bundle we have in mind.

The operators $\Delta^{\pm}(t)$ act as bounded linear operators from W^2 to L^2 and they are Fredholm as operators

$$\Delta^{\pm}(t) : W^2 \longrightarrow L^2 \; . \tag{3.3}$$

Moreover they vary continuously in the uniform operator topology (the norm topology) if their coefficients vary continuously in the uniform topology (in particular if the metric g varies continuously in C^1 topology). It follows immediately that the operators (3.3) form a real-analytic family of Fredholm operators, so we can apply Corollary 3.3. $\square$

Now we can give

Proof of Theorem 0.2. Let us consider the case of the sign $+$. Let us introduce the dependence on g explicitly in the notations for the operators $\Delta^+(t)$ and numbers $b^+(t)$, $\underline{b}^+$: denote the corresponding objects defined with the help of the Riemannian metric g by $\Delta^+(t;g)$, $b^+(t;g)$ and $\underline{b}^+(g)$.

Let us choose a Riemannian metric g_0 such that the number $\underline{b}^+(g)$ is minimal on g_0, i.e.

$$\underline{b}^+(g_0) \leq \underline{b}^+(g) \;\text{ for all metrics }\; g \; . \tag{3.4}$$

Let us introduce the following set of Riemannian metrics:

$$\mathcal{G} = \{g : \; \underline{b}^+(g) = \underline{b}^+(g_0)\}. \tag{3.5}$$

We claim that it is open and dense in C^1 topology on metrics. It follows from the upper semi-continuity of the dimension of the kernel that this set is open. In particular the equality in (3.5) is true in a neighborhood of g_0.

Let us consider a linear homotopy between g_0 and a general metric g:

$$g_s = (1-s)g_0 + sg \; , \quad s \in [0,1] \; .$$

Then $(t,s) \mapsto \Delta^+(t; g_s)$ becomes a two parameter analytic family of Fredholm operators. Obviously the background value of the dimension of the kernel for this two parameter family is equal to $\underline{b}^+(g_0)$. It follows that $\underline{b}^+(g_s) = \underline{b}^+(g_0)$ (or, equivalently, $g_s \in \mathcal{G}$) if $s \in (1-\varepsilon, 1)$ provided $\varepsilon > 0$ is sufficiently small. We see that $g = g_1$ is approximated (in C^∞ topology) by metrics $g_s \in \mathcal{G}$. Therefore $\mathcal{G}$ is dense. $\square$

Proof of Proposition 0.3. Since the operators $\Delta^{\pm}(0)$ are usual Laplacians (on $\Lambda^{\pm}(M)$), we get by the classical Hodge theorem

$$b^+(0) = \sum_k b_{2k}(M), \quad b^-(0) = \sum_k b_{2k+1}(M) \; .$$

It remains to notice that $b^{\pm}(t) \geq \underline{b}^{\pm}$ for all t due to Proposition 3.4. $\square$

Proof of Theorem 0.4. Note that the invariants $\underline{b}^{\pm}(v)$ are well defined for all vector fields (not only for the fields with non-degenerate singular points). We have seen before the formulation of Theorem 0.4 that there exists a vector field v_0 such that $\underline{b}^{\pm}(v_0)$ have the minimal possible values (0 and $|\chi(M)|$ or vice versa). Now we can apply the same argument as for the proof of Theorem 0.2. $\square$

Appendix: M.Braverman's counterexample

Here we give a counterexample (due to M.Braverman) to the Novikov conjecture ([N-S]) that the limits $b^{\pm} = \lim_{t\to\infty} b^{\pm}(t)$ do not depend on the choice of the Riemannian metric. In this counterexample in fact the limits do exist and the only possible jump is at $t = 0$. Therefore also the background values $\underline{b}^{\pm}$ do depend on the choice of the Riemannian metric.

Let us take $M = S^1 = \mathbf{R}/\mathbf{Z}$. We shall identify $C^\infty(S^1)$ with the set of all C^∞ functions on $\mathbf{R}$ which are 1-periodic (i.e. periodic with the period 1). Similarly we shall identify $\Lambda^1(S^1)$ with the set of all 1-forms $\bar{\alpha} = \alpha(x)dx$ where $\alpha \in C^\infty(\mathbf{R})$ is 1-periodic. The exterior differential $d : C^\infty(S^1) \to \Lambda^1(S^1)$ maps $f \in C^\infty(S^1)$ to $df(x) = f'(x)dx$.

We shall identify the form $\bar{\alpha}$ with the corresponding coefficient α. In this way we get an isomorphism of linear spaces $\Lambda^1(S^1) \tilde{=} C^\infty(S^1)$. Then the differential d becomes the derivative in the space of all 1-periodic C^∞ functions.

A vector field on S^1 can be written as $\bar{v} = v(x)\frac{d}{dx}$ where $v \in C^\infty(\mathbf{R})$ is 1-periodic. Then the substitution operator $i_{\bar{v}}$ is identified with the multiplication by v.

A Riemannian metric on S^1 has the form $\bar{g} = g(x)dx^2$ where $g = g_{11}$ is a 1-periodic C^∞ function. If we are given such a metric then the 1-form corresponding to the vector field $\bar{v}$ is $\bar{\omega} = \omega(x)dx$ where $\omega(x) = g(x)v(x)$.

Since $\chi(S^1) = 0$ we have $b^+(t) = b^-(t)$ for all t due to (0.9). Denote $b(t) = b^{\pm}(t)$. Similarly we shall omit the superscript $\pm$ in $\underline{b}^{\pm}$.

Proposition A1. (i) *For any vector field $\bar{v}$ on S^1 the jump of $b(t)$ can only occur at $t = 0$.*

(ii) *The jump actually occurs (and equals 1) if and only if*

$$(A1) \qquad \int_0^1 g(x)v(x)dx \neq 0.$$

.

(iii) *If the function $v(x)$ changes sign (and is not identically 0) then the background value $\underline{b}$ (which coincides with $b(t)$ for any $t \neq 0$) does depend on the metric $\bar{g}$. It equals 0 for all metrics satisfying (A1) and 1 otherwise.*

Proof. Note that $b(t) = \dim \operatorname{Ker} \partial_t^+$ where the operator $\partial_t^+ = d + t\lambda_{\bar{\omega}}$ is identified with the differential operator

$$A_t = \frac{d}{dx} + tg(x)v(x) : \ C^\infty(S^1) \longrightarrow C^\infty(S^1)\ .$$

Solving explicitly the equation $A_t\psi = 0$ in $C^\infty(\mathbf{R})$ for any $t \neq 0$ we see that a non-trivial 1-periodic solution exists if and only if the integral in the left-hand side of (A1) vanishes. In this case $b(t) = 1$ for all t, otherwise $b(t) = 0$ for all $t \neq 0$ and $b(0) = 1$. This proves (i) and (ii). Now (iii) becomes obvious. $\square$

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